Infrared Spectra of Pentachlorofluoroethane, 1, 2-Dichlorotetrafluoroethane, and 1-Bromo-2-Fluoroethane

Delia Simpson [1] and Earle K. Plyler

The infrared absorption spectra of pentachlorofluoroethane and 1,2-dichlorotetrafluoroethane have been measured from 2 to 40 μ. Several bands were found in the region from 25 to 40 μ, which had not been previously investigated. The spectrum of 1-bromo-2-fluoroethane has also been measured in the spectral region from 2 to 40 μ, and many bands were observed. A list of the observed bands and the frequency assignments for the three molecules are given.

In recent years much interest has been shown in the spectroscopy of fluorinated hydrocarbons. Glockler and Sage,[2 3] for example, have given the Raman spectra of such compounds and Smith, Nielsen, and coworkers [4 5] have given the infrared and Raman spectra of many fluorinated ethanes, along with frequency assignments for some of them. In this report we present spectra and assignments for three fluorinated hydrocarbons, pentachlorofluoroethane (CCl_3CCl_2F), 1,2-dichlorotetrafluoroethane ($CClF_2CClF_2$), and 1-bromo-2-fluoroethane (CH_2BrCH_2F). The Raman spectrum of $CClF_2CClF_2$ has been reported by Glockler and Sage (see footnote 2) and the infrared spectra of CCl_3CCl_2F and $CClF_2CClF_2$ have been reported by Smith and Nielsen (see footnotes 4 and 5). We have extended the infrared spectra of the last two named molecules to 40 μ and present the spectrum of the third in the 2 to 40 μ range.

The spectra of these compounds were obtained on a Perkin-Elmer model 12C spectrometer. The prisms used in the various spectral ranges were: 1 to 5 μ, lithium fluoride; 2 to 16 μ, sodium chloride; 14 to 24 μ, potassium bromide; and 23 to 40 μ, thallium bromide-iodide.

Figure 1 shows the spectrum of CCl_3CCl_2F. It was obtained by comparing the absorption of the compound with the absorption of air. These observations were then reduced to the percent transmittance of the compound. Figures 2 and 3 present the spectra of $CClF_2CClF_2$ and CH_2BrCH_2F obtained in a similar manner. These compounds were examined in various states and under various experimental conditions, the conditions being described on the figures and in the captions.

Because these molecules do not form a closely related series of substituted ethanes, it would not be expected that their spectra would show many similarities. A general feature of the three spectra is that most of the intense bands fall between 8 and 20 μ.

In tables 1 through 3 are given the observed infrared bands, the available Raman data, and these frequency assignments for the three molecules. These assignments were made on the basis of analogy with the assignments of closely related structures which have previously been studied. Table 4 compares these assignments for CH_2BrCH_2F with the wave number ranges for the various vibrations of three molecules. $(CH_2Cl)_2$, $(CH_2Br)_2$, and CH_2BrCH_2Cl. These values are taken from the assignments of J. U. Brown and W. S. Sheppard, who have kindly allowed the use of the results of their work, which now await publication. These four molecules listed in table 4 are expected to exist in two isomeric forms, *trans* and *gauche*. In this table are given the differences in going from the *gauche* to the *trans* form for individual vibrations. The assignments for CCl_3CCl_2F and $CClF_2CClF_2$ were made in the same way by comparing their vibrations with related molecules.

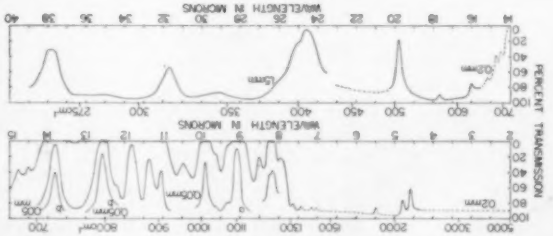

FIGURE 1. *Infrared absorption spectrum of* CCl_3CCl_2F.

The spectrum of this substance, a solid at room temperature, was obtained by making a saturated solution in CS_2. Insert A was obtained by mixing 1 part of the saturated solution with 4 parts of CS_2; insert B was obtained by mixing 1 part of the saturated solution with 20 parts of CS_2. Dotted lines represent regions of intense atmospheric absorption and of absorption of the solvent.

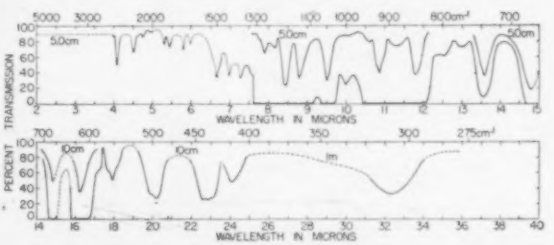

FIGURE 2. *Infrared absorption spectrum of* $CClF_2CClF_2$.

This spectrum is of the vapor state. The pressure was atmospheric, except for the inserts which were obtained by removing some of the vapor. The cell thickness employed for the different regions are marked on the figure. Dotted lines are used in regions of intense atmospheric absorption.

[1] Newnham College, Cambridge University, Cambridge, England.
[2] G. Glockler and C. Sage, J. Chem. Phys. **8**, 291 (1940).
[3] G. Glockler and C. Sage, J. Chem. Phys. **9**, 387 (1941).
[4] D. C. Smith, J. Reed Nielsen, L. H. Berryman, H. H. Claassen, and R. L. Hudson, NRL Report 3567.
[5] D. C. Smith, M. Alpert, R. A. Saunders, G. M. Brown, and N. B. Moran, NRL Report 3924.

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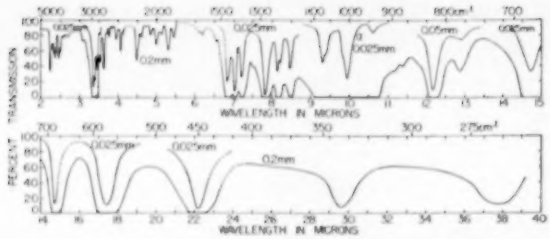

FIGURE 3. *Infrared absorption spectrum* CH_2BrCH_2F.

This spectrum is of the liquid state, except for insert A, which was obtained by diluting with CS_2 in the ratio of 1:20.

TABLE 1. *Observed infrared frequencies and band interpretations of* CCl_2CCl_2F

Two interpretations are presented, one assuming the existence of an unobserved band at 170 cm^{-1}.

Frequency	Intensity	Interpretation	Interpretation by using a 170 cm^{-1} frequency
cm^{-1}			
264	m	C—Cl	
315	m	C—Cl	
400	s	Cl—C—F 904—506	
406	s	Cl—C—F	
506	s	$C-C_\gamma$	
567	vw	880—315	400+170 406+170
625	vw	2×315	795—170
658	w	264+400	
677	w	264+406	506+170
689	w	1100—406	847—170
724	vs	$C-Cl_\gamma$	
795	vs	$C-Cl_\gamma$	
818	w	506+315 2×408	
847	s	$C-Cl_\gamma$	
880	m	$C-Cl_\gamma$	
904	m	$C-Cl_\gamma$	
954	w	----------	795+170
1009	vs	2×506	847+170
1041	w	724+315	884+170
1074	w	----------	904+170
1100	vs	$C-F_\gamma$	
1176	m	904+264	
1215	s	904+315	
1225	s	724+506	
1256	m	904+400 904+406	
1267	m	847+506	1100+170
1310	vw	904+506	
1350	vw	847+506	
1399	vw	904+506	
1418	vw	1100+315	
1825	w	1100+724	
2088	w	2×795+506 2×847+400 724+847+506	
2179	w	3×724 2×1100 2×880+400	

TABLE 2. *Observed infrared and Raman frequencies and interpretations of* $CClF_2CClF_2$

The Raman frequencies are those of Glockler and Sage (see footnote 2 of text)

Infrared frequency	Intensity	Raman frequency	Interpretation	
			Trans	*Gauche*
		171. 4	----------	C−Cl_β
		207. 4	C−Cl_β	C−Cl_β
		253. 1	C−Cl_β	
310	m	309. 5	----------	C−$F_{2\ \delta}$
		325. 7	C−F_2	
		360. 0	[$CF_{2\ r}$	
		399. 9	[$CF_{2\ w}$	
415	m	----------	$CF_{2\ r}$	
437	s	----------	$CF_{2\ w}$	
442. 5	s	443. 9	----------	$CF_{2\ \delta}$
497	s	497. 7]	----------	[$CF_{2\ r}$
503	s	503. 4]	----------	[$CF_{2\ w}$
		542. 8	$CF_{2\ t}$	
559	m	556. 5	----------	[$CF_{2\ t}$ 734−171
				[1053−498
583	w	----------	$CF_{2\ t}$	
617	vs	----------	$CF_{2\ \delta}$	
676	vs	677. 0	----------	C−C, C−Cl,
		699. 7	C−Cl,	
		723. 4	C−C,	
736	vs	734. 1		
772	w	----------	415+360 437+325	[498+310
796	w	----------	437+360	[498+310
847	vs	----------	C−Cl,	
883	m	----------	1138−253	2×443 1053−171
921	vs	914. 7	----------	CF_r
942	w	----------	617+325 360+583	498+444 503+444
1007	w	----------	----------	2×503
		1043. 5	732+325 617+437	734+310
1051	vs	1055. 9	----------	CF_r
		1063. 7	CF_r	
		1111. 6	700+415 847+253	677+444
1138	vs	----------	CF_r	
		1172. 5	CF_r	
1185]	vs	----------	CF_r	
1192]				
1227	m	----------	----------	CF_r
1267	m	1269. 6	----------	CF_r
1316	w	----------	700+617	
1357	w	----------	----------	[1053+310 2×677
				[923+444
1435	w	----------	1188+253	1268+171
1504	w	----------	[1188+325 1184+360	1053+444
			[1064+437	----------
1675	w	----------	1064+417 1138+543	1227+444
1727	w	----------	1118+543	[1053+677 1227+495
				[1227+503
1842	w	----------	1138+700	2×921
1887	w	----------	1188+700	
2110	vw	----------	----------	2×1053
2222	w			
2459	w	----------	----------	2×1227

TABLE 3. *Observed infrared frequencies and interpretations of* CH_2BrCH_2F

Frequency	Intensity	Interpretation	
		Trans	*Gauche*
cm^{-1}			
264	m	$C-X_4$	
338	m	----------	$C-X_4$
451	s	----------	$C-X_3$
574	s	----------	$C-Cl_r$
679	s	$C-Cl_r$	
776	w	CH_2 r (rocking)	
822	s	----------	CH_2 r
847	w	----------	1182−338
879	vw		
897	vw	----------	2×451
943	s	----------	CH_2 r
1006	vs	CH_2 r	$C-F_r$
1020	m	----------	$C-C_r$
1060	m	$C-C_r$	
1075	s	$C-F_r$	
1182	s	----------	CH_2 t
1218	w	CH_2 t	
1229	s	CH_2 w	
1279	vs	CH_2 t	CH_2 w
1333	w	CH_2 w	
1389	vs	----------	CH_2 w
1422	vs	----------	CH_2 δ
1461	s	CH_2 δ	
1529	vw	----------	1182+338
1618	w	----------	1279+338
1639	w	----------	1182+451 2×822
1845	w	1060+776 1075+776	[1020+822 1006+943
1890	m	1218+679	[1389+451 1279+574
1953	w	1279+679	2×943
2008	w	[2×1006 1229+776	1389+574 943+1020
		[1333+679	1182+822 2×1006
2066	w	1006+1060 1279+776	
2123	vw	2×1060	
2172	w	----------	1182+943
2226	m	1006+1218 1006+1229	1006+1182
2428	w	2×1218	1279+943
			1422+1006
2458	m	1461+1006 2×1229	1279+1182
2534	w	1461+1075	
2611	vw	1333+1279	1422+1182
2660	w	2×1333	1389+1279
2725	s	1461+1279	
2756	m	1333+1461	
2832	m	----------	2×1422
2894	s	2×1461	
2949	s	CH_r 4 modes	CH_r 4 modes
2976	s		
3125	w		
3246	w	2976+264	
3731	vw	2949+776	
3817	vw	----------	
3861	vw	----------	2949+822
3921	vw	----------	2976+943
4013	w	2949+1060	
4058	m	2976+1075	
4235	w	2949+1279	2949+1279
4290	m	2949+1333	
4329	w	----------	2949+1389
4354	m	----------	2976+1389

TABLE 4. *Comparison of the frequencies below 1,500 cm^{-1} of* $(CH_2Cl)_2$, $(CH_2Br)_2$, *and* CH_2BrCH_2Cl

The values listed for X-Y are the range of values for these three compounds, those listed after Br-F are the values for CH_2BrCH_2F. Raman values are included where available. "Not observed" means no infrared values have been reported.

Frequencies below 1,500 cm^{-1}					
CH_2			CH_2 wag		
Trans	A_g, A'	B_u, A'	*Trans*	A_g, A'	B_u, A'
X-Y Br-F	1438 to 1448 1461	1439 to 1453 1461	X-Y Br-F	1250 to 1301 1333	1192 to 1231 1229
Gauche	*A*, *A*	*B*, *A*	*Gauche*	*A*, *A*	*B*, *A*
X-Y Br-F	1419 to 1433 1422	1419 to 1433 1422	X-Y Br-F	1278 to 1312 1389	1248 to 1283 1279
CH_2 twist			CH_2 rock		
Trans	A_u, A''	B_g, A''	*Trans*	A_u, A''	B_g, A''
X-Y Br-F	1095 to 1124 1218	1250 to 1265 1279	X-Y Br-F	745 to 773 776	933 to 992 1006
Gauche	*A*, *A*	*B*, *A*	*Gauche*	*A*, *A*	*B*, *A*
X-Y Br-F	Not observed Not observed	1105 to 1145 1182	X-Y Br-F	899 to 944 943	834 to 880 822
C—Halogen,			C—Halogen		
Trans	A_g, A'	B_u, A'	*Trans*	A_g, A'	B_u, A'
X-Y Br-F	657 to 750 1075	586 to 711 679	X-Y Br-F	190 to 302 264	Not observed. Not observed.
Gauche	*A*, *A*	*B*, *A*	*Gauche*	*A*, *A*	*B*, *A*
X-Y Br-F	551 to 665 1006	583 to 678 574	X-Y Br-F	272 to 327 338	355 to 412 451
C—C,			Torsion		
Trans	A_g, A'		*Trans*	A_u, A'	
X-Y Br-F	934 to 992 1006		X-Y Br-F	Not observed. Not observed.	
Gauche	*A*, *A*		*Gauche*	*A*, *A*	
X-Y Br-F	894 to 943 943		X-Y Br-F	Not observed. Not observed.	

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Modified 1913 Reference Tables for Iron-Constantan Thermocouples

Robert J. Corruccini and Henry Shenker

An investigation was made of the characteristics of iron-constantan thermocouples typical of those being supplied to reproduce the standard temperature-electromotive force relationship that was established commercially in 1913. Using the calibration of the thermocouple that most nearly matched the 1913 reference table over the range to which iron-constantan thermocouples usually are limited (32° to 1,400° F), a slightly different temperature-electromotive force relationship was derived which is more nearly realizable by commercially available materials. The new relationship is presented in four mutually consistent tables covering the range, −320° to +1,600° F (−196° to +871° C), in both Fahrenheit and Celsius units and with both temperature and electromotive force as arguments.

1. Introduction

The application of thermocouples to pyrometry has been accompanied by the development of indicating instruments (millivoltmeters, potentiometers) with scales calibrated so as to indicate temperature directly. Such scale calibration requires a knowledge of the temperature-emf relationship of the associated thermocouple. In general, the thermocouple is required to be readily replaceable, whereas the associated indicator is relatively long-lived. It is not practical to alter the scale of the indicator to allow for changes in the characteristics of the thermocouples used with it. Consequently, each manufacturer of a given type of thermoelectric pyrometer calibrates the scales of the indicator in accordance with a definite temperature-emf relationship (reference table) and undertakes to furnish replacement thermocouples that satisfy this relationship within certain narrow limits.

Several reference tables have existed in the past for each of the common thermocouples, each supplier adopting, for a given type of thermocouple, a table (or sometimes two) that was thought to represent closely the thermocouples supplied by that company. Thus, a partial survey in 1937 showed the existence of four reference tables for iron-constantan thermocouples, differing from each other by as much as 7 percent.

In the years 1933 to 1938, the National Bureau of Standards published reference tables for all of the common thermocouples [1, 2, 3].[1,2] These tables were based on laboratory calibrations of numerous representative samples of these thermocouples. The tables in RP530 for platinum-platinum rhodium and those in RP1080 for copper-constantan were based on older tables and soon were widely accepted. The tables for chromel-alumel in RP767 likewise were widely accepted. Here, the situation was simplified by the fact that all chromel-P and alumel originates with one producer who prepares these alloys to a single set of specifications.

On the other hand, the tables for iron-constantan thermocouples published in RP1080 were the result of an investigation that began after another Government agency had noted that the NBS calibrations of iron-constantan thermocouples being purchased in the early 1930's did not match closely any existing reference table. The reference table in RP1080 represented an average of the commercially available thermocouples tested at that time and, hence, did not resemble closely any of the older tables for this type of thermocouple. RP1080 showed the manufacturers the necessity of either (1) altering the scales of new indicators, and of some existing ones, to conform to a temperature-emf relationship that was representative of the thermocouples actually being produced (RP1080), or (2) to develop new thermocouple materials that would have, within the desired tolerances, the temperature-emf relationship embodied in the scales of the existing indicators. Most of the manufacturers chose the latter course. Users with large investments in installed indicators were satisfied with this development, and the RP1080 table has not displaced the previous tables to any great extent, although it is the only table used for iron-constantan thermocouples in the field of military aircraft.

A further difference between the situations for iron-constantan and the other common thermocouples is related to the manner of procurement. Thermoelectric platinum and platinum-rhodium are prepared to specifications having the end use in view. Constantan and the purer grades of copper likewise are prepared with careful attention to the electrical properties of the product. On the other hand, such a small proportion of the total production of iron goes to thermocouples, that large producers of iron cannot be interested in controlling the properties of their product so as to suit it to this use. The manufacturer of pyrometers thus finds it more economical to select from the available production of commercial iron certain lots that happen to be suitable for thermoelectric use, rather than to have a suitable material prepared to specifications at a premium price by a producer of specialty metals. In classifying lots of iron, chemical tests are helpful, but the

[1] Figures in brackets indicate the literature references at the end of this paper.
[2] Most of the same tables are now available in expanded and revised form in NBS Circular 508, Reference Tables for Thermocouples, by Henry Shenker, John I. Lauritzen, Jr., and Robert J. Corruccini, issued May 7, 1951.

final test of the suitability of a lot must be a thermoelectric one.

In an effort to promote uniformity, steps were initiated in 1948 in the Recorder-Controller Section of the Scientific Apparatus Makers of America (SAMA), representing a number of pyrometer manufacturers, directed toward the adoption of the temperature-emf relationship known as the "1913" table[3] as a tentative standard for iron-constantan thermocouples over the range, 32° to 1,400° F. This table was chosen because it is used more than any other. It is apparently based on a single calibration of a thermocouple (no longer in existence) that was typical of the iron-constantan of forty years ago. However, experience has shown that it is apparently not possible to produce a thermocouple, using present-day commercial iron, that will correspond precisely to the 1913 table. In addition, it was desired that the standard table should extend to subzero temperatures, whereas the original 1913 table did not extend below 32° F.

It appeared that experimental data on the characteristics of thermocouples of the type that reproduce the 1913 table were required in order to (1) judge the adequacy of the 1913 table as a permanent standard for SAMA, or, if it should prove desirable, to make adjustments in the table, and (2) to obtain data at low temperatures. Accordingly, a research project for this purpose was established at the NBS and was supported in part by a grant from SAMA and in part by funds of the Bureau. This paper presents the results of this investigation.

[3] This table, sometimes called the Leeds and Northrup table, is identical with the tables given in International Critical Tables, **1**, 59, (1926) and in Tech. Pap. BS **14**, 306 (1920–21) T170, Table IV, Col. L.

2. Experimental Procedure

Each member firm of the SAMA that supplies iron-constantan thermocouples was invited to submit samples for investigation at the Bureau. These thermocouples were to be of commercial materials that were selected to match closely the 1913 table over the range 32° to 1,400° F. Of the 8 eligible member concerns, 4 responded. Of these, 3 furnished 1 lot of wire each, while the fourth concern furnished 5 different lots of wire. The samples were designated by a letter representing the supplier, followed by a number representing the sample, e. g. A–1. The wire size was approximately No. 8 AWG. Information received from some of the suppliers indicated that the iron samples came from at least four different basic sources of ingot iron.

The thermoelectric measurements were made in two parts: (a) From 32° to 1,800° F the wires were calibrated in a tube-furnace which has already been described [4, 5]. For convenience, the iron wires

TABLE 1. *Corresponding values of temperature and electromotive force for iron wires versus* Pt–27

(Reference junctions at 32° F)

Temperature (Int. 1948)	Thermal electromotive force (absolute millivolts) for samples—							
	A–1	C–1	D–1	F–1	F–3	F–4	F–5	F–6
°F								
−319.5	−2.54	−2.64	−2.54	−2.65	−2.22	−2.43	−2.35	−2.29
−297.4	−2.50	−2.60	−2.50	−2.60	−2.19	−2.40	−2.32	−2.26
−147.1	−1.65	−1.72	−1.66	−1.70	−1.49	−1.61	−1.56	−1.52
−103.0	−1.28	−1.33	−1.28	−1.31	−1.17	−1.25	−1.21	−1.18
−61.8	−0.91	−0.95	−0.91	−0.93	−0.83	−0.89	−0.86	−0.84
−27.6	−0.58	−0.61	−0.59	−0.59	−0.54	−0.58	−0.55	−0.54
+32	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100	+0.68	+0.71	+0.68	+0.68	+0.64	+0.67	+0.65	+0.64
200	1.66	1.74	1.67	1.66	1.59	1.65	1.60	1.57
300	2.59	2.72	2.60	2.56	2.50	2.57	2.50	2.46
400	3.44	3.62	3.46	3.38	3.35	3.42	3.32	3.27
500	4.20	4.42	4.23	4.09	4.13	4.18	4.06	4.00
600	4.86	5.13	4.90	4.70	4.82	4.84	4.70	4.64
700	5.45	5.76	5.49	5.25	5.45	5.43	5.28	5.21
800	5.99	6.34	6.04	5.74	6.03	5.97	5.81	5.73
900	6.52	6.90	6.57	6.23	6.60	6.49	6.32	6.24
1,000	7.06	7.47	7.12	6.74	7.17	7.03	6.86	6.77
1,100	7.66	8.09	7.72	7.31	7.80	7.63	7.45	7.36
1,200	8.34	8.79	8.41	7.97	8.51	8.31	8.12	8.04
1,300	9.14	9.60	9.21	8.76	9.33	9.11	8.92	8.83
1,400	10.07	10.52	10.15	9.68	10.25	10.04	9.84	9.76
1,500	11.06	11.44	11.17	10.73	11.21	11.08	10.85	10.76
1,600	12.01	12.26	12.19	11.83	12.11	12.19	11.86	11.71
1,700	12.77	12.98	12.94	12.76	12.89	13.20	12.70	12.61
1,800	13.57	13.79	13.75	13.61	13.71	14.00	13.51	13.44

were calibrated against a standard iron wire, and the constantan wires against a standard constantan wire. The standard iron and the standard constantan wires had previously been calibrated against Pt 27, the platinum thermoelectric standard of the NBS, and they were rechecked at frequent intervals during each run against the platinum element of the platinum versus platinum-10 percent rhodium thermocouple used to measure the temperature. The test wires were calibrated three at a time. The measuring junctions of the test wires, the standard wire of the same material, and the platinum versus platinum-rhodium thermocouple used to measure the temperature were welded together. Because of the relative instability of the iron and constantan at temperatures above 1,400° F, measurements in this range were made against a platinum standard only. The average rate of rise of temperature in the region above 1,400° F was about 8 deg F per minute. All measurements were made at 100 deg F intervals on rising temperature. After each run, the heated section of the iron or constantan standard was cut off and discarded. The methods used were similar to those described in [4, p. 262–5, 277]. The measurements are considered to be accurate within the equivalent of 1 deg F. (b) From +32° to −319.5° F, the thermal emf of each wire against a copper reference wire was determined in stirred liquid baths. The copper reference wires were calibrated against a platinum standard. Down to −147° F the cryostat described by Scott [6; 5, p. 206] was used. Observations at two lower temperatures were made by using baths of liquid oxygen and nitrogen stirred by a stream of the same gas. Temperatures in the baths were determined with a platinum resistance thermometer. Sections of wire that had not been previously heated were used for the low-temperature measurements. The limit of error in this range is considered to be the equivalent of ½ deg F.

In all the measurements, the reference junctions were kept at 32° F in a mixture of ice and water. An atmosphere of air surrounded the thermocouples at all times. The calibrations of the individual wires relative to Pt 27 at the various calibration temperatures were compiled and are given in tables 1 and 2. The data in table 3 for the complete thermocouples were obtained by combination of the data in tables 1 and 2.

The deviations (expressed in deg F) of the various thermocouples from the 1913 table are shown in figure 1. The deviation of the RP1080 table is also shown. Because the A–1 thermocouple gave the nearest over-all match to the 1913 table in the range up to 1,400° F, the calibrations of the iron and constantan elements of this thermocouple were used as base lines for figures 2 and 3, respectively, in which are shown the differences among the various individual wires. The curves marked "RP1080" repre-

TABLE 2. *Corresponding values of temperature and electromotive forces for constantan wires versus* Pt–27

(Reference junctions at 32° F)

Temperature (Int. 1948)	Thermal electromotive force (absolute millivolts) for samples—						
	A–1	C–1	D–1	F–1, F–6	F–3	F–4	F–5
°F							
−319.5	+5.24	+5.14	+5.24	+5.31	+5.24	+5.30	+5.30
−297.4	4.99	4.89	4.99	5.05	4.98	5.04	5.04
−147.1	2.96	2.90	2.95	2.99	2.95	2.99	2.99
−103.0	2.28	2.33	2.28	2.31	2.28	2.31	2.31
−61.8	1.62	1.59	1.62	1.64	1.62	1.64	1.64
−27.6	1.05	1.03	1.05	1.06	1.05	1.06	1.06
+32	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100	−1.26	−1.24	−1.27	−1.28	−1.26	−1.28	−1.28
200	−3.24	−3.17	−3.25	−3.29	−3.24	−3.28	−3.28
300	−5.35	−5.26	−5.34	−5.43	−5.35	−5.42	−5.42
400	−7.58	−7.46	−7.60	−7.69	−7.58	−7.67	−7.67
500	−9.92	−9.75	−9.93	−10.04	−9.90	−10.01	−10.02
600	−12.32	−12.10	−12.33	−12.47	−12.30	−12.43	−12.45
700	−14.80	−14.54	−14.80	−14.97	−14.76	−14.92	−14.95
800	−17.33	−17.01	−17.31	−17.51	−17.25	−17.45	−17.48
900	−19.88	−19.51	−19.85	−20.08	−19.78	−20.01	−20.05
1,000	−22.46	−22.06	−22.42	−22.68	−22.34	−22.60	−22.64
1,100	−25.07	−24.61	−25.00	−25.29	−24.91	−25.20	−25.25
1,200	−27.67	−27.18	−27.59	−27.90	−27.49	−27.80	−27.86
1,300	−30.29	−29.76	−30.18	−30.52	−30.07	−30.41	−30.48
1,400	−32.89	−32.32	−32.77	−33.13	−32.64	−33.01	−33.08
1,500	−35.48	−34.86	−35.32	−35.71	−35.16	−35.59	−35.66
1,600	−38.04	−37.28	−37.86	−38.27	−37.72	−38.15	−38.22
1,700	−40.57	−39.87	−40.36	−40.79	−40.22	−40.67	−40.74
1,800	−43.06	−42.34	−42.84	−43.29	−42.68	−43.15	−43.23

251762—53——2

TABLE 3. *Corresponding values of temperature and electromotive force for iron-constantan thermocouples*

(Reference junctions at 32° F)

Temperature (Int. 1948)	Thermal electromotive force (absolute millivolts) for samples—									
	A–1	C–1	D–1	F–1	F–3	F–4	F–5	F–6	1913 table	RP1080 table
° *F*										
−319. 5	−7. 79	−7. 79	−7. 78	−7. 96	−7. 45	−7. 73	−7. 66	−7. 60	-----	-----
−297. 4	−7. 49	−7. 49	−7. 49	−7. 65	−7. 17	−7. 44	−7. 37	−7. 31	-----	−7. 83
−147. 1	−4. 61	−4. 62	−4. 61	−4. 69	−4. 44	−4. 59	−4. 55	−4. 51	-----	−4. 80
−103. 0	−3. 56	−3. 57	−3. 56	−3. 62	−3. 44	−3. 55	−3. 52	−3. 49	-----	−3. 71
−61. 8	−2. 53	−2. 53	−2. 53	−2. 57	−2. 45	−2. 53	−2. 50	−2. 48	-----	−2. 63
−27. 6	−1. 63	−1. 63	−1. 63	−1. 65	−1. 58	−1. 63	−1. 61	−1. 60	-----	−1. 69
+32	0. 00	0. 00	0. 00	0. 00	0. 00	0. 00	0. 00	0. 00	0. 00	0. 00
100	+1. 94	+1. 94	+1. 95	+1. 96	+1. 90	+1. 95	+1. 93	+1. 92	+1. 96	+2. 00
200	4. 91	4. 91	4. 92	4. 95	4. 83	4. 93	4. 88	4. 86	4. 92	5. 03
300	7. 94	7. 97	7. 97	7. 99	7. 85	7. 99	7. 91	7. 88	7. 95	8. 13
400	11. 03	11. 07	11. 07	11. 06	10. 93	11. 09	11. 00	10. 96	11. 02	11. 25
500	14. 12	14. 17	14. 16	14. 13	14. 03	14. 19	14. 08	14. 04	14. 10	14. 35
600	17. 18	17. 23	17. 23	17. 17	17. 12	17. 27	17. 15	17. 10	17. 17	17. 43
700	20. 26	20. 30	20. 30	20. 22	20. 21	20. 35	20. 23	20. 18	20. 24	20. 49
800	23. 32	23. 35	23. 35	23. 25	23. 29	23. 42	23. 29	23. 24	23. 32	23. 54
900	26. 40	26. 41	26. 42	26. 31	26. 38	26. 50	26. 37	26. 32	26. 42	26. 60
1000	29. 52	29. 53	29. 54	29. 41	29. 51	29. 63	29. 50	29. 45	29. 55	29. 71
1100	32. 72	32. 70	32. 72	32. 60	32. 72	32. 82	32. 70	32. 65	32. 75	32. 89
1200	36. 01	35. 97	36. 00	35. 88	36. 00	36. 11	35. 98	35. 94	36. 05	36. 18
1300	39. 43	39. 36	39. 40	39. 28	39. 40	39. 52	39. 40	39. 36	39. 46	39. 58
1400	42. 96	42. 84	42. 92	42. 81	42. 89	43. 05	42. 92	42. 89	42. 95	43. 11
1500	46. 53	46. 30	46. 49	46. 44	46. 37	46. 67	46. 51	46. 48	46. 47	46. 73
1600	50. 05	49. 64	50. 05	50. 10	49. 83	50. 34	50. 08	49. 98	49. 99	50. 38
1700	53. 34	52. 86	53. 31	53. 55	53. 10	53. 87	53. 44	53. 40	53. 52	53. 86
1800	56. 64	56. 12	56. 59	56. 91	56. 38	57. 16	56. 74	56. 73	57. 04	57. 17

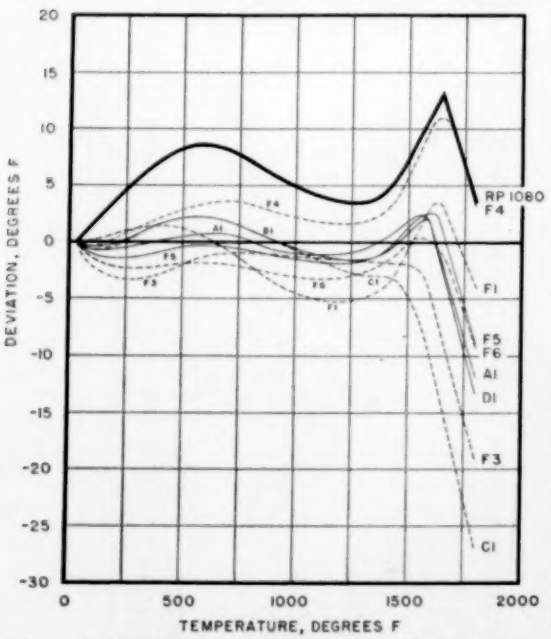

FIGURE 1. *Deviation of SAMA iron-constantan thermocouples and RP1080 table from the 1913 table.*

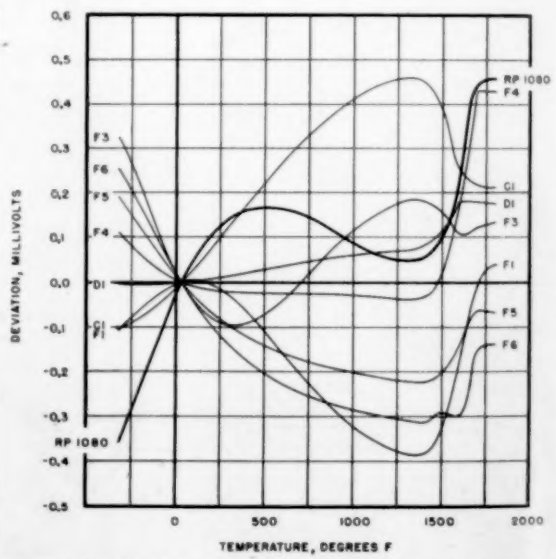

FIGURE 2. *Deviation of SAMA iron samples and RP1080 iron from iron sample A–1.*

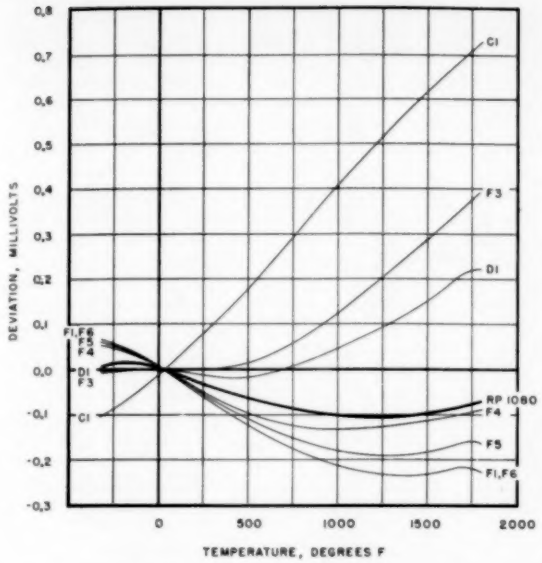

FIGURE 3. *Deviation of SAMA constantan samples and RP1080 constantan from constantan sample A–1.*

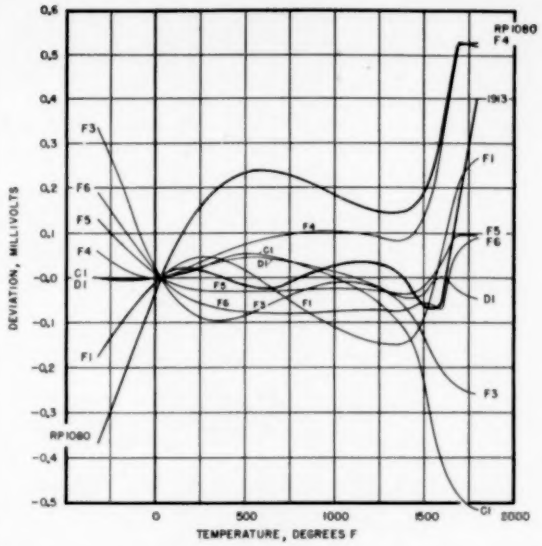

FIGURE 4. *Deviations of SAMA iron-constantan thermocouples, the 1913 table, and the RP1080 table from thermocouple A–1.*

sent the data for iron versus platinum and constantan versus platinum given in [3]. The 1913 table was for iron versus constantan only and, consequently, the corresponding data for the individual elements versus platinum are not available for inclusion in figures 2 and 3. The constantan marked "F–1, F–6" was a single sample intended for use in combination with either the F–1 or F–6 iron sample. In figure 4 similar plots for the completed thermocouples are given. In preparing these figures, the 1913 and RP1080 tables were adjusted to the emf units and temperature scale used in this investigation and currently in use at this Bureau. These are the absolute system of electrical units [7] and the International Temperature Scale of 1948 [8]. One absolute volt is equal to 0.99967 international volt (U. S.). The differences between the International Temperature Scales of 1948 and 1927 are not more than 0.8 deg F in the range of this investigation [9] and so are nearly negligible.

The differences in shape of the various curves in figure 4 are mainly due to the iron elements because the deviations of the constantan elements from each other are approximately linear. It is well known that thermal emf depends both on chemical composition and on the degree of strain. In hope of gaining some insight into the differences among the various thermocouples, the iron samples were analyzed chemically, and their hardnesses were determined. This work was done by the Chemistry and Metallurgy Divisions of the Bureau. The results are given in tables 4 and 5.

TABLE 4. *Chemical and spectrographic analyses of iron samples*

[Values are given in percent]

Sample	C	Mn	P	S	Cu	Si*	Ni*	Cr*	Sn*
A–1	0.07	0.25	0.015	0.031	0.12	<0.01	0.044	0.015	<0.01
C–1	.11	.38	.005	.031	.054	<.01	.030	.016	<.01
D–1	.04	.29	.009	.024	.14	<.01	.042	.016	<.01
F–1	.02	.03	.004	.029	.15	<.01	.039	.025	<.01
F–3	.08	.43	.080	.028	.018	.02	.011	<.01	.01
F–4	.02	.23	.011	.016	.023	<.01	.014	<.01	<.01
F–5	.06	.21	.006	.020	.022	.01	.013	<.01	<.01
F–6	.05	.22	.006	.018	.022	<.01	.012	<.01	.01

*Spectographic determination.

TABLE 5. *Results of hardness measurements on samples of iron wire*

Sample	Vickers number*	Equivalent Rockwell B number
A–1	99. 8	55
C–1	106	59. 5
D–1	138	76
F–1	127	71
F–3	210	94
F–4	83. 6	39
F–5	138	76
F–6	131	73

*Determinations were made with a 10-kg load applied through a square-based diamond-pyramid indenter.

3. Discussion

The sharp changes in the slopes of the various curves in figure 1 at about 1,650° F are thought to be due to the 1913 table having been based in this region on an extrapolation that neglected the effect of the abrupt change in thermoelectric power of iron at the $\alpha-\gamma$ transition. At lower temperatures the various curves show a characteristic undulation. It is not known to what extent this is due to errors in the original 1913 calibration and to what extent to differences between the chemical compositions characteristic of iron used for thermocouples in 1913 and at present.

It was hardly to be expected that the curves of figure 2 would be fully interpretable, inasmuch as the research was designed to test these commercial materials in the "as received" condition, and did not provide for that control of the variables of chemical composition and heat treatment that would be necessary for a fundamental understanding of the results.

The quantum theory of metals leads to the correct predictions that iron is thermoelectrically positive to platinum and that the effect of small percentages of copper on the thermoelectric power is opposite in sign to the effect of manganese. However, a detailed explanation of the features shown in figure 2 is beyond the present scope of theory. Nevertheless, a few generalizations may be noted from the data.

Roeser and Dahl [3] showed that the 1913 table could be approximated by using iron wires having several tenths percent of Mn, whereas the samples of which RP1080 was characteristic contained less than 0.1 percent of Mn. The effects of small percentages of the minor elements that are common in commercial iron on the thermoelectric power of iron have been investigated by Finch [10], who described several compositions of iron that can be used in thermocouples. Finch showed that the thermoelectric power of iron at 500° C (932° F) is increased by additions of Cr, S, or Mn and decreased by Sn, Si, P, Ni, and (in percentages greater than 0.10) Cu, whereas the effect of carbon is negligible. It will be noted that the curve for iron sample F–1 is similar in shape to that for RP1080 iron, and, indeed its manganese content of only 0.03 percent marks it as an RP1080-type iron. In the other iron samples, manganese is the principal minor element, and the thermal emf at 932° F (see fig. 2), with minor exceptions, increases with the manganese concentration, as would be predicted from the work of Finch.

The bunching of the curves in the gamma region above 1,650° F may be the result of differences in electronic band-structure in alpha and gamma iron. However, the steep rise of F–1 and F–4 is exceptional. These samples were lowest in carbon (0.02 percent), whereas the samples showing the opposite behavior, C–1 and F–3, were highest and second highest, respectively, in carbon content (0.11 and 0.08 percent). The effect thus may be connected with the solution of carbon in the gamma-phase iron. A study of the phases present initially was not made, and so no positive statement may be made on this point.

Dahl has shown [11; 5, p. 1238] that thermocouple iron decreases in thermoelectric power on heating at temperatures above 1,200° F. However, the annealing of high-purity iron increases the thermoelectric power, so that it appears that the drift in the thermal emf of the iron elements of thermocouples at high temperatures is primarily due to phase changes and chemical changes, and that relief of strain is not an important factor in this drift.

The unique shape of curve F–3 may be related to any of the following factors: (1) Its manganese content was highest of all (0.43 percent); (2) Its phosphorus content was very much higher than that of the other samples; (3) Its hardness was unusually high.

The curves in figure 3 show the approximately linear deviation that is characteristic of constantan [3].

The maximum differences among the iron samples and also among the constantan samples amounted to about 0.8 mv at 1,400° F, which is equivalent to about 20 deg F in the indication of an iron-constantan thermocouple. In addition, there were marked differences in the curve shape for the iron samples. Nevertheless, the calibrations of the completed thermocouples fell within a satisfactorily narrow band up to 1,500° F. The width of this band on either side of the relation for thermocouple A–1 was approximately equivalent to commercial tolerances for the iron-constantan thermocouple, thus indicating the practicality of the A–1 relation for a commercial standard. However, the large differences among the wires of a given kind illustrate the impracticality of establishing standards for the emf of the individual elements (against platinum) as has been done for the elements of all the other common thermocouples.

The deviations of the thermocouples below 32° F are quite large in terms of temperature, which indicates the necessity for care in establishing the calibrations of thermocouples in this range. However, the deviations in emf are nearly linear, so that relatively few calibration temperatures need be used.

4. Reference Table for 1913-type Iron-Constantan Thermocouples

As a result of this investigation, the Thermocouple Calibration Curves Committee, Recorder-Controller Section, SAMA, has recommended adoption of the temperature electromotive force relationship of thermocouple A–1 as a tentative standard. Because of the known instability of iron-constantan thermocouples above 1,600° F, and also because of the large increase in deviation of the thermocouples from each other in this region, the upper limit of the standard has been set at 1,600° F. (Manufacturers of this thermocouple do not recommend its use above 1,400° to 1,500° F, except in special circumstances.) The differences between the A–1 relationship and the 1913 table, when both are expressed in consistent electrical units and temperature scale, are given in table 6.

TABLE 6. *Differences between A–1 and 1913 tables for iron-constantan thermocouples*

Temperature	Emf A–1 table minus emf 1913 table	
°F	*Millivolts*	*Equivalent degrees F*
32	0.00	0.0
100	−.02	−.7
200	−.01	−.3
300	−.01	−.3
400	+.01	+.3
500	+.02	+.7
600	+.01	+.3
700	+.02	+.7
800	.00	.0
900	−.02	−.7
1,000	−.03	−.9
1,100	−.03	−.9
1,200	−.04	−1.2
1,300	−.03	−0.9
1,400	+.01	+.3
1,500	+.06	+1.7
1,600	+.06	+1.7

In December 1950, preliminary results of this investigation were summarized in a published article [12]. In this article it was pointed out that the temperature-emf relationship of sample A–1 was favored for the proposed new SAMA standard. In order to determine whether or not the A–1 relationship would be acceptable as an industry-wide standard, replacing not merely the 1913 table but also the RP1080 table, answers to the following questions were solicited: (1) Is industry-wide standardization on a single reference table for iron-constantan desirable? (2) Is the 1913 table, or a close approximation to it, acceptable for this purpose? (3) What degree of hardship will be occasioned by abandonment of the RP1080 table?

The replies received were uniformly favorable to a single industry-wide standard. Users and suppliers of 1913-type thermocouples were uniformly favorable to the proposed SAMA standard. However, most of the replies received from users and suppliers of solely RP1080-type thermocouples indicated that it would not be practicable for them to change to the proposed SAMA standard. It is important to point out that the change from RP1080-type to A–1-type thermocouples would require alterations to most of the associated indicating equipment, whereas the difference between the 1913 and A–1 relationships is much smaller and is less than commercial tolerances for this thermocouple.

Thus it appears that at least two standards, the SAMA table (A–1) and the RP1080 table, will coexist in the immediate future. Most commercial production of iron-constantan will probably be based on the SAMA table, as it has been based on the 1913 table in the past. On the other hand, one of the principal fields in which the RP1080 standard is dominant—namely, in piston-engined military aircraft—is declining in importance.

In order to make the A–1 relationship useful, it has been expanded into four tables, numbers 7, 8, 9, and 10. Before doing this, additional samples of the A lot were calibrated as a check against gross errors. The differences between these and the calibration of A–1 were within the limit of error of the calibration process. In table 7, the desired function for the region from −320° to 0° F was obtained at intervals of 1 deg by the method of divided differences. For the range from 0° to +1600° F, the LaGrange polynomial was used. Linear interpolation, either direct or inverse, was then performed on this table to obtain the remaining tables. The preparation of tables 7 to 10 was carried out by the Applied Mathematics Division of the Bureau.

The thermoelectric measurements below 32° F were made in the Thermometry Laboratory by Ruth C. Gallman and Eugenia Thomas. It is a pleasure to acknowledge the advice given by Wm. F. Roeser and also the cooperation of the various members of the Thermocouple Calibration Curves Committee of SAMA and their chairman, Edward J. Albert.

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Table 7. Iron-Constantan Thermocouples

(Emf in Absolute Millivolts. Temperatures in Degrees F.* Reference Junctions at 32° F.)

°F	0	1	2	3	4	5	6	7	8	9	10	°F
	Millivolts											
-310	-7.66	-7.68	-7.69	-7.70	-7.71	-7.73	-7.74	-7.75	-7.76	-7.78	-7.79	-310
-300	-7.52	-7.54	-7.55	-7.57	-7.58	-7.59	-7.61	-7.62	-7.64	-7.65	-7.66	-300
-290	-7.38	-7.39	-7.40	-7.42	-7.44	-7.45	-7.46	-7.48	-7.49	-7.51	-7.52	-290
-280	-7.22	-7.24	-7.25	-7.27	-7.28	-7.30	-7.31	-7.33	-7.34	-7.36	-7.38	-280
-270	-7.06	-7.07	-7.09	-7.11	-7.12	-7.14	-7.15	-7.17	-7.19	-7.20	-7.22	-270
-260	-6.89	-6.90	-6.92	-6.94	-6.96	-6.97	-6.99	-7.01	-7.02	-7.04	-7.06	-260
-250	-6.71	-6.73	-6.75	-6.77	-6.78	-6.80	-6.82	-6.84	-6.85	-6.87	-6.89	-250
-240	-6.53	-6.55	-6.57	-6.59	-6.61	-6.62	-6.64	-6.66	-6.68	-6.70	-6.71	-240
-230	-6.35	-6.37	-6.38	-6.40	-6.42	-6.44	-6.46	-6.48	-6.50	-6.52	-6.53	-230
-220	-6.16	-6.18	-6.19	-6.21	-6.23	-6.25	-6.27	-6.29	-6.31	-6.33	-6.35	-220
-210	-5.96	-5.98	-6.00	-6.02	-6.04	-6.06	-6.08	-6.10	-6.12	-6.14	-6.16	-210
-200	-5.76	-5.78	-5.80	-5.82	-5.84	-5.86	-5.88	-5.90	-5.92	-5.94	-5.96	-200
-190	-5.55	-5.57	-5.59	-5.61	-5.63	-5.65	-5.67	-5.70	-5.72	-5.74	-5.76	-190
-180	-5.34	-5.36	-5.38	-5.40	-5.42	-5.44	-5.46	-5.49	-5.51	-5.53	-5.55	-180
-170	-5.12	-5.14	-5.16	-5.19	-5.21	-5.23	-5.25	-5.27	-5.30	-5.32	-5.34	-170
-160	-4.90	-4.92	-4.94	-4.97	-4.99	-5.01	-5.03	-5.06	-5.08	-5.10	-5.12	-160
-150	-4.68	-4.70	-4.72	-4.74	-4.76	-4.79	-4.81	-4.83	-4.86	-4.88	-4.90	-150
-140	-4.44	-4.47	-4.49	-4.51	-4.54	-4.56	-4.58	-4.61	-4.63	-4.65	-4.68	-140
-130	-4.21	-4.23	-4.26	-4.28	-4.30	-4.33	-4.35	-4.38	-4.40	-4.42	-4.44	-130
-120	-3.97	-4.00	-4.02	-4.04	-4.07	-4.09	-4.12	-4.14	-4.16	-4.19	-4.21	-120
-110	-3.73	-3.76	-3.78	-3.81	-3.83	-3.85	-3.88	-3.90	-3.93	-3.95	-3.97	-110
-100	-3.49	-3.51	-3.54	-3.56	-3.59	-3.61	-3.64	-3.66	-3.68	-3.71	-3.73	-100
-90	-3.24	-3.27	-3.29	-3.32	-3.34	-3.36	-3.39	-3.41	-3.44	-3.46	-3.49	-90
-80	-2.99	-3.02	-3.04	-3.07	-3.09	-3.12	-3.14	-3.17	-3.19	-3.22	-3.24	-80
-70	-2.74	-2.76	-2.79	-2.81	-2.84	-2.86	-2.89	-2.92	-2.94	-2.97	-2.99	-70
-60	-2.48	-2.51	-2.53	-2.56	-2.58	-2.61	-2.64	-2.66	-2.69	-2.71	-2.74	-60
-50	-2.22	-2.25	-2.27	-2.30	-2.33	-2.35	-2.38	-2.40	-2.43	-2.46	-2.48	-50
-40	-1.96	-1.99	-2.01	-2.04	-2.06	-2.09	-2.12	-2.14	-2.17	-2.20	-2.22	-40
-30	-1.70	-1.72	-1.75	-1.78	-1.80	-1.83	-1.86	-1.88	-1.91	-1.94	-1.96	-30
-20	-1.43	-1.46	-1.48	-1.51	-1.54	-1.56	-1.59	-1.62	-1.64	-1.67	-1.70	-20
-10	-1.16	-1.19	-1.21	-1.24	-1.27	-1.29	-1.32	-1.35	-1.38	-1.40	-1.43	-10
(-)0	-.89	-.91	-.94	-.97	-1.00	-1.02	-1.05	-1.08	-1.10	-1.13	-1.16	(-)0
(+)0	-.89	-.86	-.83	-.80	-.78	-.75	-.72	-.70	-.67	-.64	-.61	(+)0

*Based on the International Temperature Scale of 1948.

Table 7. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees F.* Reference Junctions at 32° F.)

°F	0	1	2	3	4	5	6	7	8	9	10	°F
	Millivolts											
(+)0	-.89	-.86	-.83	-.80	-.78	-.75	-.72	-.70	-.67	-.64	-.61	(+)0
10	-.61	-.58	-.56	-.53	-.50	-.48	-.45	-.42	-.39	-.36	-.34	10
20	-.34	-.31	-.28	-.25	-.22	-.20	-.17	-.14	-.11	-.09	-.06	20
30	-.06	-.03	-.00	.03	.05	.08	.11	.14	.17	.19	.22	30
40	.22	.25	.28	.31	.34	.36	.39	.42	.45	.48	.50	40
50	.50	.53	.56	.59	.62	.65	.67	.70	.73	.76	.79	50
60	.79	.82	.84	.87	.90	.93	.96	.99	1.02	1.04	1.07	60
70	1.07	1.10	1.13	1.16	1.19	1.22	1.25	1.28	1.30	1.33	1.36	70
80	1.36	1.39	1.42	1.45	1.48	1.51	1.54	1.56	1.59	1.62	1.65	80
90	1.65	1.68	1.71	1.74	1.77	1.80	1.83	1.85	1.88	1.91	1.94	90
100	1.94	1.97	2.00	2.03	2.06	2.09	2.12	2.14	2.17	2.20	2.23	100
110	2.23	2.26	2.29	2.32	2.35	2.38	2.41	2.44	2.47	2.50	2.52	110
120	2.52	2.55	2.58	2.61	2.64	2.67	2.70	2.73	2.76	2.79	2.82	120
130	2.82	2.85	2.88	2.91	2.94	2.97	3.00	3.03	3.06	3.08	3.11	130
140	3.11	3.14	3.17	3.20	3.23	3.26	3.29	3.32	3.35	3.38	3.41	140
150	3.41	3.44	3.47	3.50	3.53	3.56	3.59	3.62	3.65	3.68	3.71	150
160	3.71	3.74	3.77	3.80	3.83	3.86	3.89	3.92	3.95	3.98	4.01	160
170	4.01	4.04	4.07	4.10	4.13	4.16	4.19	4.22	4.25	4.28	4.31	170
180	4.31	4.34	4.37	4.40	4.43	4.46	4.49	4.52	4.55	4.58	4.61	180
190	4.61	4.64	4.67	4.70	4.73	4.76	4.79	4.82	4.85	4.88	4.91	190
200	4.91	4.94	4.97	5.00	5.03	5.06	5.09	5.12	5.15	5.18	5.21	200
210	5.21	5.24	5.27	5.30	5.33	5.36	5.39	5.42	5.45	5.48	5.51	210
220	5.51	5.54	5.57	5.60	5.63	5.66	5.69	5.72	5.75	5.78	5.81	220
230	5.81	5.84	5.87	5.90	5.93	5.96	5.99	6.02	6.05	6.08	6.11	230
240	6.11	6.14	6.17	6.20	6.24	6.27	6.30	6.33	6.36	6.39	6.42	240
250	6.42	6.45	6.48	6.51	6.54	6.57	6.60	6.63	6.66	6.69	6.72	250
260	6.72	6.75	6.78	6.81	6.84	6.87	6.90	6.93	6.96	7.00	7.03	260
270	7.03	7.06	7.09	7.12	7.15	7.18	7.21	7.24	7.27	7.30	7.33	270
280	7.33	7.36	7.39	7.42	7.45	7.48	7.51	7.54	7.58	7.61	7.64	280
290	7.64	7.67	7.70	7.73	7.76	7.79	7.82	7.85	7.88	7.91	7.94	290
300	7.94	7.97	8.00	8.04	8.07	8.10	8.13	8.16	8.19	8.22	8.25	300

*Based on the International Temperature Scale of 1948.

Table 7. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees F.* Reference Junctions at 32° F.)

°F	0	1	2	3	4	5	6	7	8	9	10	°F
	Millivolts											
300	7.94	7.97	8.00	8.04	8.07	8.10	8.13	8.16	8.19	8.22	8.25	300
310	8.25	8.28	8.31	8.34	8.37	8.40	8.44	8.47	8.50	8.53	8.56	310
320	8.56	8.59	8.62	8.65	8.68	8.71	8.74	8.77	8.80	8.84	8.87	320
330	8.87	8.90	8.93	8.96	8.99	9.02	9.05	9.08	9.11	9.14	9.17	330
340	9.17	9.20	9.24	9.27	9.30	9.33	9.36	9.39	9.42	9.45	9.48	340
350	9.48	9.51	9.54	9.58	9.61	9.64	9.67	9.70	9.73	9.76	9.79	350
360	9.79	9.82	9.85	9.88	9.92	9.95	9.98	10.01	10.04	10.07	10.10	360
370	10.10	10.13	10.16	10.19	10.22	10.25	10.28	10.32	10.35	10.38	10.41	370
380	10.41	10.44	10.47	10.50	10.53	10.56	10.60	10.63	10.66	10.69	10.72	380
390	10.72	10.75	10.78	10.81	10.84	10.87	10.90	10.94	10.97	11.00	11.03	390
400	11.03	11.06	11.09	11.12	11.15	11.18	11.21	11.24	11.28	11.31	11.34	400
410	11.34	11.37	11.40	11.43	11.46	11.49	11.52	11.55	11.58	11.62	11.65	410
420	11.65	11.68	11.71	11.74	11.77	11.80	11.83	11.86	11.89	11.92	11.96	420
430	11.96	11.99	12.02	12.05	12.08	12.11	12.14	12.17	12.20	12.23	12.26	430
440	12.26	12.30	12.33	12.36	12.39	12.42	12.45	12.48	12.51	12.54	12.57	440
450	12.57	12.60	12.64	12.67	12.70	12.73	12.76	12.79	12.82	12.85	12.88	450
460	12.88	12.91	12.94	12.98	13.01	13.04	13.07	13.10	13.13	13.16	13.19	460
470	13.19	13.22	13.25	13.28	13.31	13.34	13.38	13.41	13.44	13.47	13.50	470
480	13.50	13.53	13.56	13.59	13.62	13.65	13.68	13.72	13.75	13.78	13.81	480
490	13.81	13.84	13.87	13.90	13.93	13.96	13.99	14.02	14.05	14.08	14.12	490
500	14.12	14.15	14.18	14.21	14.24	14.27	14.30	14.33	14.36	14.39	14.42	500
510	14.42	14.45	14.48	14.52	14.55	14.58	14.61	14.64	14.67	14.70	14.73	510
520	14.73	14.76	14.79	14.82	14.85	14.88	14.91	14.94	14.98	15.01	15.04	520
530	15.04	15.07	15.10	15.13	15.16	15.19	15.22	15.25	15.28	15.31	15.34	530
540	15.34	15.37	15.40	15.44	15.47	15.50	15.53	15.56	15.59	15.62	15.65	540
550	15.65	15.68	15.71	15.74	15.77	15.80	15.84	15.87	15.90	15.93	15.96	550
560	15.96	15.99	16.02	16.05	16.08	16.11	16.14	16.17	16.20	16.23	16.26	560
570	16.26	16.30	16.33	16.36	16.39	16.42	16.45	16.48	16.51	16.54	16.57	570
580	16.57	16.60	16.63	16.66	16.69	16.72	16.75	16.78	16.82	16.85	16.88	580
590	16.88	16.91	16.94	16.97	17.00	17.03	17.06	17.09	17.12	17.15	17.18	590
600	17.18	17.21	17.24	17.28	17.31	17.34	17.37	17.40	17.43	17.46	17.49	600

* Based on the International Temperature Scale of 1948.

Table 7. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees F.* Reference Junctions at 32° F.)

°F	0	1	2	3	4	5	6	7	8	9	10	°F
	Millivolts											
600	17.18	17.21	17.24	17.28	17.31	17.34	17.37	17.40	17.43	17.46	17.49	600
610	17.49	17.52	17.55	17.58	17.61	17.64	17.68	17.71	17.74	17.77	17.80	610
620	17.80	17.83	17.86	17.89	17.92	17.95	17.98	18.01	18.04	18.08	18.11	620
630	18.11	18.14	18.17	18.20	18.23	18.26	18.29	18.32	18.35	18.38	18.41	630
640	18.41	18.44	18.47	18.50	18.54	18.57	18.60	18.63	18.66	18.69	18.72	640
650	18.72	18.75	18.78	18.81	18.84	18.87	18.90	18.94	18.97	19.00	19.03	650
660	19.03	19.06	19.09	19.12	19.15	19.18	19.21	19.24	19.27	19.30	19.34	660
670	19.34	19.37	19.40	19.43	19.46	19.49	19.52	19.55	19.58	19.61	19.64	670
680	19.64	19.67	19.70	19.74	19.77	19.80	19.83	19.86	19.89	19.92	19.95	680
690	19.95	19.98	20.01	20.04	20.07	20.10	20.13	20.16	20.20	20.23	20.26	690
700	20.26	20.29	20.32	20.35	20.38	20.41	20.44	20.47	20.50	20.53	20.56	700
710	20.56	20.59	20.62	20.66	20.69	20.72	20.75	20.78	20.81	20.84	20.87	710
720	20.87	20.90	20.93	20.96	20.99	21.02	21.05	21.08	21.11	21.14	21.18	720
730	21.18	21.21	21.24	21.27	21.30	21.33	21.36	21.39	21.42	21.45	21.48	730
740	21.48	21.51	21.54	21.57	21.60	21.64	21.67	21.70	21.73	21.76	21.79	740
750	21.79	21.82	21.85	21.88	21.91	21.94	21.97	22.00	22.03	22.06	22.10	750
760	22.10	22.13	22.16	22.19	22.22	22.25	22.28	22.31	22.34	22.37	22.40	760
770	22.40	22.43	22.46	22.49	22.52	22.55	22.58	22.62	22.65	22.68	22.71	770
780	22.71	22.74	22.77	22.80	22.83	22.86	22.89	22.92	22.95	22.98	23.01	780
790	23.01	23.04	23.08	23.11	23.14	23.17	23.20	23.23	23.26	23.29	23.32	790
800	23.32	23.35	23.38	23.41	23.44	23.47	23.50	23.53	23.56	23.60	23.63	800
810	23.63	23.66	23.69	23.72	23.75	23.78	23.81	23.84	23.87	23.90	23.93	810
820	23.93	23.96	23.99	24.02	24.06	24.09	24.12	24.15	24.18	24.21	24.24	820
830	24.24	24.27	24.30	24.33	24.36	24.39	24.42	24.45	24.48	24.52	24.55	830
840	24.55	24.58	24.61	24.64	24.67	24.70	24.73	24.76	24.79	24.82	24.85	840
850	24.85	24.88	24.91	24.94	24.98	25.01	25.04	25.07	25.10	25.13	25.16	850
860	25.16	25.19	25.22	25.25	25.28	25.32	25.35	25.38	25.41	25.44	25.47	860
870	25.47	25.50	25.53	25.56	25.59	25.62	25.65	25.68	25.72	25.75	25.78	870
880	25.78	25.81	25.84	25.87	25.90	25.93	25.96	25.99	26.02	26.06	26.09	880
890	26.09	26.12	26.15	26.18	26.21	26.24	26.27	26.30	26.33	26.36	26.40	890
900	26.40	26.43	26.46	26.49	26.52	26.55	26.58	26.61	26.64	26.67	26.70	900

* Based on the International Temperature Scale of 1948.

Table 7. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts Temperatures in Degrees F.* Reference Junctions at 32° F.)

°F	0	1	2	3	4	5	6	7	8	9	10	°F
	Millivolts											
900	26.40	26.43	26.46	26.49	26.52	26.55	26.58	26.61	26.64	26.67	26.70	900
910	26.70	26.74	26.77	26.80	26.83	26.86	26.89	26.92	26.95	26.98	27.02	910
920	27.02	27.05	27.08	27.11	27.14	27.17	27.20	27.23	27.26	27.30	27.33	920
930	27.33	27.36	27.39	27.42	27.45	27.48	27.51	27.54	27.58	27.61	27.64	930
940	27.64	27.67	27.70	27.73	27.76	27.80	27.83	27.86	27.89	27.92	27.95	940
950	27.95	27.98	28.02	28.05	28.08	28.11	28.14	28.17	28.20	28.23	28.26	950
960	28.26	28.30	28.33	28.36	28.39	28.42	28.45	28.48	28.52	28.55	28.58	960
970	28.58	28.61	28.64	28.67	28.70	28.74	28.77	28.80	28.83	28.86	28.89	970
980	28.89	28.92	28.96	28.99	29.02	29.05	29.08	29.11	29.14	29.18	29.21	980
990	29.21	29.24	29.27	29.30	29.33	29.37	29.40	29.43	29.46	29.49	29.52	990
1000	29.52	29.56	29.59	29.62	29.65	29.68	29.71	29.75	29.78	29.81	29.84	1000
1010	29.84	29.87	29.90	29.94	29.97	30.00	30.03	30.06	30.10	30.13	30.16	1010
1020	30.16	30.19	30.22	30.25	30.28	30.32	30.35	30.38	30.41	30.44	30.48	1020
1030	30.48	30.51	30.54	30.57	30.60	30.64	30.67	30.70	30.73	30.76	30.80	1030
1040	30.80	30.83	30.86	30.89	30.92	30.96	30.99	31.02	31.05	31.08	31.12	1040
1050	31.12	31.15	31.18	31.21	31.24	31.28	31.31	31.34	31.37	31.40	31.44	1050
1060	31.44	31.47	31.50	31.53	31.56	31.60	31.63	31.66	31.69	31.72	31.76	1060
1070	31.76	31.79	31.82	31.85	31.88	31.92	31.95	31.98	32.01	32.05	32.08	1070
1080	32.08	32.11	32.14	32.18	32.21	32.24	32.27	32.30	32.34	32.37	32.40	1080
1090	32.40	32.43	32.47	32.50	32.53	32.56	32.60	32.63	32.66	32.69	32.72	1090
1100	32.72	32.76	32.79	32.82	32.86	32.89	32.92	32.95	32.98	33.02	33.05	1100
1110	33.05	33.08	33.11	33.15	33.18	33.21	33.24	33.28	33.31	33.34	33.37	1110
1120	33.37	33.41	33.44	33.47	33.50	33.54	33.57	33.60	33.64	33.67	33.70	1120
1130	33.70	33.73	33.76	33.80	33.83	33.86	33.90	33.93	33.96	33.99	34.03	1130
1140	34.03	34.06	34.09	34.12	34.16	34.19	34.22	34.26	34.29	34.32	34.36	1140
1150	34.36	34.39	34.42	34.45	34.49	34.52	34.55	34.58	34.62	34.65	34.68	1150
1160	34.68	34.72	34.75	34.78	34.82	34.85	34.88	34.92	34.95	34.98	35.01	1160
1170	35.01	35.05	35.08	35.11	35.15	35.18	35.21	35.25	35.28	35.31	35.35	1170
1180	35.35	35.38	35.41	35.45	35.48	35.51	35.54	35.58	35.61	35.64	35.68	1180
1190	35.68	35.71	35.74	35.78	35.81	35.84	35.88	35.91	35.94	35.98	36.01	1190
1200	36.01	36.05	36.08	36.11	36.15	36.18	36.21	36.25	36.28	36.31	36.35	1200

*Based on the International Temperature Scale of 1948.

Table 7. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees F.* Reference Junctions at 32° F.)

°F	0	1	2	3	4	5	6	7	8	9	10	°F
	Millivolts											
1200	36.01	36.05	36.08	36.11	36.15	36.18	36.21	36.25	36.28	36.31	36.35	1200
1210	36.35	36.38	36.42	36.45	36.48	36.52	36.55	36.58	36.62	36.65	36.69	1210
1220	36.69	36.72	36.75	36.79	36.82	36.86	36.89	36.92	36.96	36.99	37.02	1220
1230	37.02	37.06	37.09	37.13	37.16	37.20	37.23	37.26	37.30	37.33	37.36	1230
1240	37.36	37.40	37.43	37.47	37.50	37.54	37.57	37.60	37.64	37.67	37.71	1240
1250	37.71	37.74	37.78	37.81	37.84	37.88	37.91	37.95	37.98	38.02	38.05	1250
1260	38.05	38.08	38.12	38.15	38.19	38.22	38.26	38.29	38.32	38.36	38.39	1260
1270	38.39	38.43	38.46	38.50	38.53	38.57	38.60	38.64	38.67	38.70	38.74	1270
1280	38.74	38.77	38.81	38.84	38.88	38.91	38.95	38.98	39.02	39.05	39.08	1280
1290	39.08	39.12	39.15	39.19	39.22	39.26	39.29	39.33	39.36	39.40	39.43	1290
1300	39.43	39.47	39.50	39.54	39.57	39.61	39.64	39.68	39.71	39.75	39.78	1300
1310	39.78	39.82	39.85	39.89	39.92	39.96	39.99	40.03	40.06	40.10	40.13	1310
1320	40.13	40.17	40.20	40.24	40.27	40.31	40.34	40.38	40.41	40.45	40.48	1320
1330	40.48	40.52	40.55	40.59	40.62	40.66	40.69	40.73	40.76	40.80	40.83	1330
1340	40.83	40.87	40.90	40.94	40.98	41.01	41.05	41.08	41.12	41.15	41.19	1340
1350	41.19	41.22	41.26	41.29	41.33	41.36	41.40	41.43	41.47	41.50	41.54	1350
1360	41.54	41.58	41.61	41.65	41.68	41.72	41.75	41.79	41.82	41.86	41.90	1360
1370	41.90	41.93	41.97	42.00	42.04	42.07	42.11	42.14	42.18	42.22	42.25	1370
1380	42.25	42.29	42.32	42.36	42.39	42.43	42.46	42.50	42.53	42.57	42.61	1380
1390	42.61	42.64	42.68	42.71	42.75	42.78	42.82	42.85	42.89	42.92	42.96	1390
1400	42.96	43.00	43.03	43.07	43.10	43.14	43.18	43.21	43.25	43.28	43.32	1400
1410	43.32	43.35	43.39	43.43	43.46	43.50	43.53	43.57	43.60	43.64	43.68	1410
1420	43.68	43.71	43.75	43.78	43.82	43.85	43.89	43.92	43.96	44.00	44.03	1420
1430	44.03	44.07	44.10	44.14	44.18	44.21	44.25	44.28	44.32	44.35	44.39	1430
1440	44.39	44.42	44.46	44.50	44.53	44.57	44.60	44.64	44.68	44.71	44.75	1440
1450	44.75	44.78	44.82	44.85	44.89	44.93	44.96	45.00	45.03	45.07	45.10	1450
1460	45.10	45.14	45.18	45.21	45.25	45.28	45.32	45.35	45.39	45.42	45.46	1460
1470	45.46	45.50	45.53	45.57	45.60	45.64	45.68	45.71	45.75	45.78	45.82	1470
1480	45.82	45.85	45.89	45.92	45.96	46.00	46.03	46.07	46.10	46.14	46.18	1480
1490	46.18	46.21	46.25	46.28	46.32	46.35	46.39	46.42	46.46	46.50	46.53	1490
1500	46.53	46.57	46.60	46.64	46.67	46.71	46.74	46.78	46.82	46.85	46.89	1500
1510	46.89	46.92	46.96	47.00	47.03	47.07	47.10	47.14	47.17	47.21	47.24	1510
1520	47.24	47.28	47.32	47.35	47.39	47.42	47.46	47.49	47.53	47.56	47.60	1520
1530	47.60	47.63	47.67	47.70	47.74	47.78	47.81	47.85	47.88	47.92	47.95	1530
1540	47.95	47.99	48.02	48.06	48.09	48.13	48.16	48.20	48.24	48.27	48.31	1540
1550	48.31	48.34	48.38	48.41	48.45	48.48	48.52	48.55	48.59	48.62	48.66	1550
1560	48.66	48.69	48.73	48.76	48.80	48.83	48.87	48.90	48.94	48.97	49.01	1560
1570	49.01	49.04	49.08	49.11	49.15	49.18	49.22	49.25	49.29	49.32	49.36	1570
1580	49.36	49.39	49.43	49.46	49.50	49.53	49.56	49.60	49.63	49.67	49.70	1580
1590	49.70	49.74	49.77	49.81	49.84	49.88	49.91	49.94	49.98	50.01	50.05	1590
1600	50.05											1600

*Based on the International Temperature Scale of 1948.

Table 8. Iron-Constantan Thermocouples

(Emf in Absolute Millivolts. Temperatures in Degrees F*. Reference Junctions at 32° F)

Millivolts	.00	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00	Millivolts
	Degrees F											
-7.00	-267	-273	-279	-285	-292	-298	-305	-313	-320			-7.00
-6.00	-212	-217	-222	-228	-233	-238	-244	-249	-255	-261	-267	-6.00
-5.00	-165	-169	-174	-178	-183	-188	-192	-197	-202	-207	-212	-5.00
-4.00	-121	-125	-130	-134	-138	-142	-147	-151	-156	-160	-165	-4.00
-3.00	-80	-84	-88	-92	-96	-100	-105	-109	-113	-117	-121	-3.00
-2.00	-42	-45	-49	-53	-57	-61	-65	-69	-72	-76	-80	-2.00
-1.00	-4	-8	-12	-15	-19	-23	-26	-30	-34	-38	-42	-1.00
(-)0.00	32	29	25	21	18	14	10	7	3	0	-4	(-)0.00
(+)0.00	32	36	39	43	46	50	53	57	60	64	67	(+)0.00
1.00	67	71	74	78	81	85	88	92	95	99	102	1.00
2.00	102	105	109	112	116	119	123	126	129	133	136	2.00
3.00	136	140	143	146	150	153	156	160	163	166	170	3.00
4.00	170	173	177	180	183	187	190	193	196	200	203	4.00
5.00	203	206	210	213	216	220	223	226	230	233	236	5.00
6.00	236	240	243	246	249	253	256	259	263	266	269	6.00
7.00	269	272	276	279	282	286	289	292	295	299	302	7.00
8.00	302	305	308	312	315	318	321	325	328	331	334	8.00
9.00	334	338	341	344	347	351	354	357	360	364	367	9.00
10.00	367	370	373	376	380	383	386	389	393	396	399	10.00
11.00	399	402	406	409	412	415	419	422	425	428	431	11.00
12.00	431	435	438	441	444	448	451	454	457	461	464	12.00
13.00	464	467	470	474	477	480	483	487	490	493	496	13.00
14.00	496	499	503	506	509	513	516	519	522	526	529	14.00
15.00	529	532	535	539	542	545	548	552	555	558	561	15.00
16.00	561	565	568	571	574	578	581	584	587	591	594	16.00
17.00	594	597	601	604	607	610	614	617	620	623	627	17.00
18.00	627	630	633	636	640	643	646	649	653	656	659	18.00
19.00	659	662	666	669	672	675	679	682	685	688	692	19.00
20.00	692	695	698	701	705	708	711	714	718	721	724	20.00

*Based on the International Temperature Scale of 1948.

Table 8. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees F*. Reference Junctions at 32° F)

Millivolts	.00	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00	Millivolts
	Degrees F											
20.00	692	695	698	701	705	708	711	714	718	721	724	20.00
21.00	724	727	731	734	737	741	744	747	750	754	757	21.00
22.00	757	760	763	767	770	773	777	780	783	786	790	22.00
23.00	790	793	796	799	803	806	809	812	816	819	822	23.00
24.00	822	825	829	832	835	839	842	845	848	852	855	24.00
25.00	855	858	861	865	868	871	874	878	881	884	887	25.00
26.00	887	890	894	897	900	903	907	910	913	916	919	26.00
27.00	919	923	926	929	932	936	939	942	945	948	952	27.00
28.00	952	955	958	961	964	967	971	974	977	980	983	28.00
29.00	983	987	990	993	996	999	1002	1006	1009	1012	1015	29.00
30.00	1015	1018	1021	1024	1028	1031	1034	1037	1040	1043	1046	30.00
31.00	1046	1050	1053	1056	1059	1062	1065	1068	1071	1074	1078	31.00
32.00	1078	1081	1084	1087	1090	1093	1096	1099	1102	1105	1108	32.00
33.00	1108	1112	1115	1118	1121	1124	1127	1130	1133	1136	1139	33.00
34.00	1139	1142	1145	1148	1151	1154	1157	1160	1164	1167	1170	34.00
35.00	1170	1173	1176	1179	1182	1185	1188	1191	1194	1197	1200	35.00
36.00	1200	1203	1206	1209	1212	1215	1217	1220	1223	1226	1229	36.00
37.00	1229	1232	1235	1238	1241	1244	1247	1250	1253	1256	1259	37.00
38.00	1259	1261	1264	1267	1270	1273	1276	1279	1282	1285	1288	38.00
39.00	1288	1290	1293	1296	1299	1302	1305	1308	1311	1313	1316	39.00
40.00	1316	1319	1322	1325	1328	1331	1333	1336	1339	1342	1345	40.00
41.00	1345	1348	1350	1353	1356	1359	1362	1365	1367	1370	1373	41.00
42.00	1373	1376	1379	1381	1384	1387	1390	1393	1395	1398	1401	42.00
43.00	1401	1404	1407	1410	1412	1415	1418	1421	1424	1426	1429	43.00
44.00	1429	1432	1435	1438	1440	1443	1446	1449	1451	1454	1457	44.00
45.00	1457	1460	1463	1465	1468	1471	1474	1477	1480	1482	1485	45.00
46.00	1485	1488	1491	1494	1496	1499	1502	1505	1508	1510	1513	46.00
47.00	1513	1516	1519	1522	1524	1527	1530	1533	1536	1539	1541	47.00
48.00	1541	1544	1547	1550	1553	1556	1558	1561	1564	1567	1570	48.00
49.00	1570	1573	1576	1578	1581	1584	1587	1590	1593	1596	1599	49.00
50.00	1599	1602										50.00

*Based on the International Temperature Scale of 1948.

Table 9. Iron-Constantan Thermocouples

(Emf in Absolute Millivolts. Temperatures in Degrees C(Int.1948) Reference Junctions at 0° C)

°C	0	1	2	3	4	5	6	7	8	9	10	°C
	Millivolts											
-190	-7.66	-7.69	-7.71	-7.73	-7.76	-7.78						-190
-180	-7.40	-7.43	-7.46	-7.49	-7.51	-7.54	-7.56	-7.59	-7.61	-7.64	-7.66	-180
-170	-7.12	-7.15	-7.18	-7.21	-7.24	-7.27	-7.30	-7.32	-7.35	-7.38	-7.40	-170
-160	-6.82	-6.85	-6.88	-6.91	-6.94	-6.97	-7.00	-7.03	-7.06	-7.09	-7.12	-160
-150	-6.50	-6.53	-6.56	-6.60	-6.63	-6.66	-6.69	-6.72	-6.76	-6.79	-6.82	-150
-140	-6.16	-6.19	-6.22	-6.26	-6.29	-6.33	-6.36	-6.40	-6.43	-6.46	-6.50	-140
-130	-5.80	-5.84	-5.87	-5.91	-5.94	-5.98	-6.01	-6.05	-6.08	-6.12	-6.16	-130
-120	-5.42	-5.46	-5.50	-5.54	-5.58	-5.61	-5.65	-5.69	-5.72	-5.76	-5.80	-120
-110	-5.03	-5.07	-5.11	-5.15	-5.19	-5.23	-5.27	-5.31	-5.35	-5.38	-5.42	-110
-100	-4.63	-4.67	-4.71	-4.75	-4.79	-4.83	-4.87	-4.91	-4.95	-4.99	-5.03	-100
-90	-4.21	-4.25	-4.30	-4.34	-4.38	-4.42	-4.46	-4.50	-4.55	-4.59	-4.63	-90
-80	-3.78	-3.82	-3.87	-3.91	-3.96	-4.00	-4.04	-4.08	-4.13	-4.17	-4.21	-80
-70	-3.34	-3.38	-3.43	-3.47	-3.52	-3.56	-3.60	-3.65	-3.69	-3.74	-3.78	-70
-60	-2.89	-2.94	-2.98	-3.03	-3.07	-3.12	-3.16	-3.21	-3.25	-3.30	-3.34	-60
-50	-2.43	-2.48	-2.52	-2.57	-2.62	-2.66	-2.71	-2.75	-2.80	-2.84	-2.89	-50
-40	-1.96	-2.01	-2.06	-2.10	-2.15	-2.20	-2.24	-2.29	-2.34	-2.38	-2.43	-40
-30	-1.48	-1.53	-1.58	-1.63	-1.67	-1.72	-1.77	-1.82	-1.87	-1.91	-1.96	-30
-20	-1.00	-1.04	-1.09	-1.14	-1.19	-1.24	-1.29	-1.34	-1.39	-1.43	-1.48	-20
-10	-0.50	-0.55	-0.60	-0.65	-0.70	-0.75	-0.80	-0.85	-0.90	-0.95	-1.00	-10
(-)0	0.00	-0.05	-0.10	-0.15	-0.20	-0.25	-0.30	-0.35	-0.40	-0.45	-0.50	(-)0
(+)0	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	(+)0
10	0.50	0.56	0.61	0.66	0.71	0.76	0.81	0.86	0.91	0.97	1.02	10
20	1.02	1.07	1.12	1.17	1.22	1.28	1.33	1.38	1.43	1.48	1.54	20
30	1.54	1.59	1.64	1.69	1.74	1.80	1.85	1.90	1.95	2.00	2.06	30
40	2.06	2.11	2.16	2.22	2.27	2.32	2.37	2.42	2.48	2.53	2.58	40
50	2.58	2.64	2.69	2.74	2.80	2.85	2.90	2.96	3.01	3.06	3.11	50
60	3.11	3.17	3.22	3.27	3.33	3.38	3.43	3.49	3.54	3.60	3.65	60
70	3.65	3.70	3.76	3.81	3.86	3.92	3.97	4.02	4.08	4.13	4.19	70
80	4.19	4.24	4.29	4.35	4.40	4.46	4.51	4.56	4.62	4.67	4.73	80
90	4.73	4.78	4.83	4.89	4.94	5.00	5.05	5.10	5.16	5.21	5.27	90
100	5.27	5.32	5.38	5.43	5.48	5.54	5.59	5.65	5.70	5.76	5.81	100

Table 9. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees C(Int.1948) Reference Junctions at 0° C)

°C	0	1	2	3	4	5	6	7	8	9	10	°C
	Millivolts											
100	5.27	5.32	5.38	5.43	5.48	5.54	5.59	5.65	5.70	5.76	5.81	100
110	5.81	5.86	5.92	5.97	6.03	6.08	6.14	6.19	6.25	6.30	6.36	110
120	6.36	6.41	6.47	6.52	6.58	6.63	6.68	6.74	6.79	6.85	6.90	120
130	6.90	6.96	7.01	7.07	7.12	7.18	7.23	7.29	7.34	7.40	7.45	130
140	7.45	7.51	7.56	7.62	7.67	7.73	7.78	7.84	7.89	7.95	8.00	140
150	8.00	8.06	8.12	8.17	8.23	8.28	8.34	8.39	8.45	8.50	8.56	150
160	8.56	8.61	8.67	8.72	8.78	8.84	8.89	8.95	9.00	9.06	9.11	160
170	9.11	9.17	9.22	9.28	9.33	9.39	9.44	9.50	9.56	9.61	9.67	170
180	9.67	9.72	9.78	9.83	9.89	9.95	10.00	10.06	10.11	10.17	10.22	180
190	10.22	10.28	10.34	10.39	10.45	10.50	10.56	10.61	10.67	10.72	10.78	190
200	10.78	10.84	10.89	10.95	11.00	11.06	11.12	11.17	11.23	11.28	11.34	200
210	11.34	11.39	11.45	11.50	11.56	11.62	11.67	11.73	11.78	11.84	11.89	210
220	11.89	11.95	12.00	12.06	12.12	12.17	12.23	12.28	12.34	12.39	12.45	220
230	12.45	12.50	12.56	12.62	12.67	12.73	12.78	12.84	12.89	12.95	13.01	230
240	13.01	13.06	13.12	13.17	13.23	13.28	13.34	13.40	13.45	13.51	13.56	240
250	13.56	13.62	13.67	13.73	13.78	13.84	13.89	13.95	14.00	14.06	14.12	250
260	14.12	14.17	14.23	14.28	14.34	14.39	14.45	14.50	14.56	14.61	14.67	260
270	14.67	14.72	14.78	14.83	14.89	14.94	15.00	15.06	15.11	15.17	15.22	270
280	15.22	15.28	15.33	15.39	15.44	15.50	15.55	15.61	15.66	15.72	15.77	280
290	15.77	15.83	15.88	15.94	16.00	16.05	16.11	16.16	16.22	16.27	16.33	290
300	16.33	16.38	16.44	16.49	16.55	16.60	16.66	16.71	16.77	16.82	16.88	300
310	16.88	16.93	16.99	17.04	17.10	17.15	17.21	17.26	17.32	17.37	17.43	310
320	17.43	17.48	17.54	17.60	17.65	17.71	17.76	17.82	17.87	17.93	17.98	320
330	17.98	18.04	18.09	18.15	18.20	18.26	18.32	18.37	18.43	18.48	18.54	330
340	18.54	18.59	18.65	18.70	18.76	18.81	18.87	18.92	18.98	19.03	19.09	340
350	19.09	19.14	19.20	19.26	19.31	19.37	19.42	19.48	19.53	19.59	19.64	350
360	19.64	19.70	19.75	19.81	19.86	19.92	19.97	20.03	20.08	20.14	20.20	360
370	20.20	20.25	20.31	20.36	20.42	20.47	20.53	20.58	20.64	20.69	20.75	370
380	20.75	20.80	20.86	20.91	20.97	21.02	21.08	21.13	21.19	21.24	21.30	380
390	21.30	21.35	21.41	21.46	21.52	21.57	21.63	21.68	21.74	21.79	21.85	390
400	21.85	21.90	21.96	22.02	22.07	22.13	22.18	22.24	22.29	22.35	22.40	400

Table 9. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees C(Int.1948) Reference Junctions at 0° C)

°C	0	1	2	3	4	5	6	7	8	9	10	°C
	Millivolts											
400	21.85	21.90	21.96	22.02	22.07	22.13	22.18	22.24	22.29	22.35	22.40	400
410	22.40	22.46	22.51	22.57	22.62	22.68	22.73	22.79	22.84	22.90	22.95	410
420	22.95	23.01	23.06	23.12	23.17	23.23	23.28	23.34	23.39	23.45	23.50	420
430	23.50	23.56	23.61	23.67	23.72	23.78	23.83	23.89	23.94	24.00	24.06	430
440	24.06	24.11	24.17	24.22	24.28	24.33	24.39	24.44	24.50	24.55	24.61	440
450	24.61	24.66	24.72	24.77	24.83	24.88	24.94	25.00	25.05	25.11	25.16	450
460	25.16	25.22	25.27	25.33	25.38	25.44	25.49	25.55	25.60	25.66	25.72	460
470	25.72	25.77	25.83	25.88	25.94	25.99	26.05	26.10	26.16	26.22	26.27	470
480	26.27	26.33	26.38	26.44	26.49	26.55	26.61	26.66	26.72	26.77	26.83	480
490	26.83	26.89	26.94	27.00	27.05	27.11	27.17	27.22	27.28	27.33	27.39	490
500	27.39	27.45	27.50	27.56	27.61	27.67	27.73	27.78	27.84	27.90	27.95	500
510	27.95	28.01	28.07	28.12	28.18	28.23	28.29	28.35	28.40	28.46	28.52	510
520	28.52	28.57	28.63	28.69	28.74	28.80	28.86	28.91	28.97	29.02	29.08	520
530	29.08	29.14	29.20	29.25	29.31	29.37	29.42	29.48	29.54	29.59	29.65	530
540	29.65	29.71	29.76	29.82	29.88	29.94	29.99	30.05	30.11	30.16	30.22	540
550	30.22	30.28	30.34	30.39	30.45	30.51	30.57	30.62	30.68	30.74	30.80	550
560	30.80	30.85	30.91	30.97	31.02	31.08	31.14	31.20	31.26	31.31	31.37	560
570	31.37	31.43	31.49	31.54	31.60	31.66	31.72	31.78	31.83	31.89	31.95	570
580	31.95	32.01	32.06	32.12	32.18	32.24	32.30	32.36	32.41	32.47	32.53	580
590	32.53	32.59	32.65	32.71	32.76	32.82	32.88	32.94	33.00	33.06	33.11	590
600	33.11	33.17	33.23	33.29	33.35	33.41	33.46	33.52	33.58	33.64	33.70	600
610	33.70	33.76	33.82	33.88	33.94	33.99	34.05	34.11	34.17	34.23	34.29	610
620	34.29	34.35	34.41	34.47	34.53	34.58	34.64	34.70	34.76	34.82	34.88	620
630	34.88	34.94	35.00	35.06	35.12	35.18	35.24	35.30	35.36	35.42	35.48	630
640	35.48	35.54	35.60	35.66	35.72	35.78	35.84	35.90	35.96	36.02	36.08	640
650	36.08	36.14	36.20	36.26	36.32	36.38	36.44	36.50	36.56	36.62	36.69	650
660	36.69	36.75	36.81	36.87	36.93	36.99	37.05	37.11	37.18	37.24	37.30	660
670	37.30	37.36	37.42	37.48	37.54	37.60	37.66	37.73	37.79	37.85	37.91	670
680	37.91	37.97	38.04	38.10	38.16	38.22	38.28	38.34	38.41	38.47	38.53	680
690	38.53	38.59	38.66	38.72	38.78	38.84	38.90	38.97	39.03	39.09	39.15	690
700	39.15	39.22	39.28	39.34	39.40	39.47	39.53	39.59	39.65	39.72	39.78	700
710	39.78	39.84	39.91	39.97	40.03	40.10	40.16	40.22	40.28	40.35	40.41	710
720	40.41	40.48	40.54	40.60	40.66	40.73	40.79	40.86	40.92	40.98	41.05	720
730	41.05	41.11	41.17	41.24	41.30	41.36	41.43	41.49	41.56	41.62	41.68	730
740	41.68	41.75	41.81	41.87	41.94	42.00	42.07	42.13	42.19	42.26	42.32	740
750	42.32	42.38	42.45	42.51	42.58	42.64	42.70	42.77	42.83	42.90	42.96	750
760	42.96	43.02	43.09	43.15	43.22	43.28	43.35	43.41	43.48	43.54	43.60	760
770	43.60	43.67	43.73	43.80	43.86	43.92	43.99	44.05	44.12	44.18	44.25	770
780	44.25	44.31	44.38	44.44	44.50	44.57	44.63	44.70	44.76	44.82	44.89	780
790	44.89	44.95	45.02	45.08	45.15	45.21	45.28	45.34	45.40	45.47	45.53	790
800	45.53	45.60	45.66	45.72	45.79	45.85	45.92	45.98	46.05	46.11	46.18	800
810	46.18	46.24	46.30	46.37	46.43	46.50	46.56	46.62	46.69	46.75	46.82	810
820	46.82	46.88	46.94	47.01	47.07	47.14	47.20	47.27	47.33	47.39	47.46	820
830	47.46	47.52	47.58	47.65	47.71	47.78	47.84	47.90	47.97	48.03	48.09	830
840	48.09	48.16	48.22	48.28	48.35	48.41	48.48	48.54	48.60	48.66	48.73	840
850	48.73	48.79	48.85	48.92	48.98	49.04	49.10	49.17	49.23	49.29	49.36	850
860	49.36	49.42	49.48	49.54	49.61	49.67	49.73	49.79	49.86	49.92	49.98	860
870	49.98	50.04										870

Table 10. Iron-Constantan Thermocouples

(Emf in Absolute Millivolts. Temperatures in Degrees C(Int.1948) Reference Junctions at 0° C)

Millivolts	.00	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00	Millivolts
	Degrees C											
-7.00	-166	-169	-173	-176	-180	-184	-188	-192	-196			-7.00
-6.00	-136	-138	-141	-144	-147	-150	-153	-156	-159	-163	-166	-6.00
-5.00	-109	-112	-114	-117	-119	-122	-125	-127	-130	-133	-136	-5.00
-4.00	-85	-87	-90	-92	-94	-97	-99	-102	-104	-107	-109	-4.00
-3.00	-62	-65	-67	-69	-71	-74	-76	-78	-80	-83	-85	-3.00
-2.00	-41	-43	-45	-47	-49	-52	-54	-56	-58	-60	-62	-2.00
-1.00	-20	-22	-24	-26	-28	-30	-32	-35	-37	-39	-41	-1.00
(-)0.00	0	-2	-4	-6	-8	-10	-12	-14	-16	-18	-20	(-)0.00
(+)0.00	0	2	4	6	8	10	12	14	16	18	20	(+)0.00
1.00	20	22	24	25	27	29	31	33	35	37	39	1.00
2.00	39	41	43	45	47	48	50	52	54	56	58	2.00
3.00	58	60	62	63	65	67	69	71	73	75	77	3.00
4.00	77	78	80	82	84	86	88	90	91	93	95	4.00
5.00	95	97	99	101	102	104	106	108	110	112	113	5.00
6.00	113	115	117	119	121	123	124	126	128	130	132	6.00
7.00	132	134	135	137	139	141	143	144	146	148	150	7.00
8.00	150	152	154	155	157	159	161	163	164	166	168	8.00
9.00	168	170	172	173	175	177	179	181	182	184	186	9.00
10.00	186	188	190	191	193	195	197	199	200	202	204	10.00
11.00	204	206	208	209	211	213	215	217	218	220	222	11.00
12.00	222	224	226	227	229	231	233	235	236	238	240	12.00
13.00	240	242	244	245	247	249	251	253	254	256	258	13.00
14.00	258	260	262	263	265	267	269	271	272	274	276	14.00
15.00	276	278	280	281	283	285	287	289	290	292	294	15.00
16.00	294	296	298	300	301	303	305	307	309	310	312	16.00
17.00	312	314	316	318	319	321	323	325	327	329	330	17.00
18.00	330	332	334	336	338	339	341	343	345	347	348	18.00
19.00	348	350	352	354	356	357	359	361	363	365	366	19.00
20.00	366	368	370	372	374	376	377	379	381	383	385	20.00

Table 10. Iron-Constantan Thermocouples—Continued

(Emf in Absolute Millivolts. Temperatures in Degrees C(Int.1948) Reference Junctions at 0° C)

Millivolts	.00	.10	.20	.30	.40	.50	.60	.70	.80	.90	1.00	Millivolts
	Degrees C											
20.00	366	368	370	372	374	376	377	379	381	383	385	20.00
21.00	385	386	388	390	392	394	395	397	399	401	403	21.00
22.00	403	405	406	408	410	412	414	416	417	419	421	22.00
23.00	421	423	424	426	428	430	432	434	435	437	439	23.00
24.00	439	441	443	444	446	448	450	452	453	455	457	24.00
25.00	457	459	461	463	464	466	468	470	472	473	475	25.00
26.00	475	477	479	481	482	484	486	488	489	491	493	26.00
27.00	493	495	497	499	500	502	504	506	507	509	511	27.00
28.00	511	513	514	516	518	520	522	523	525	527	529	28.00
29.00	529	530	532	534	536	537	539	541	543	544	546	29.00
30.00	546	548	550	551	553	555	557	558	560	562	564	30.00
31.00	564	565	567	569	571	572	574	576	577	579	581	31.00
32.00	581	583	584	586	588	589	591	593	595	596	598	32.00
33.00	598	600	601	603	605	607	608	610	612	613	615	33.00
34.00	615	617	618	620	622	624	625	627	629	630	632	34.00
35.00	632	634	635	637	639	640	642	644	645	647	649	35.00
36.00	649	650	652	654	655	657	659	660	662	664	665	36.00
37.00	665	667	668	670	672	673	675	677	678	680	681	37.00
38.00	681	683	685	686	688	690	691	693	694	696	698	38.00
39.00	698	699	701	702	704	706	707	709	710	712	713	39.00
40.00	713	715	717	718	720	721	723	725	726	728	729	40.00
41.00	729	731	732	734	736	737	739	740	742	743	745	41.00
42.00	745	747	748	750	751	753	754	756	757	759	761	42.00
43.00	761	762	764	765	767	768	770	772	773	775	776	43.00
44.00	776	778	779	781	782	784	785	787	789	790	792	44.00
45.00	792	793	795	796	798	799	801	803	804	806	807	45.00
46.00	807	809	810	812	814	815	817	818	820	821	823	46.00
47.00	823	824	826	828	829	831	832	834	835	837	839	47.00
48.00	839	840	842	843	845	846	848	850	851	853	854	48.00
49.00	854	856	858	859	861	862	864	866	867	869	870	49.00
50.00	870	872										50.00

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Epitaxial Deposits of Metals Evaporated on Salt Substrates[1]

Olive G. Engel

Experimental data and theoretical considerations relating to the process of epitaxy are reviewed. The case of epitaxy resulting from vapor phase deposition of metals on rock salt is considered in detail and it is concluded that a factor other than best fit determines the orientation. A model for epitaxy in metal–rock salt pairs is advanced which involves ionization of the metal atoms on the surface. The epitaxial temperature is found to be related to the ionization potential of the metal for the ionization state to be expected. It is found that the observed orientations of metals on rock salt are those that would be expected if such an ionization process occurred, that is, if the orientation involved charged ions rather than neutral atoms.

1. Introduction

The increasing importance that is being attached to thin films has been pointed out by Holland [1],[2] and a number of applications have been cited by Lewis [2], who arbitrarily defines "thin" to mean of the order of 100 A. Recently the Electron Physics Section of the National Bureau of Standards has needed films that are not only thin by this criterion but that also retain the perfect orientation of a single crystal. Films of this kind were required both for the electron interferometer of Marton [3] and for experiments on the single inelastic scattering of electrons.

For the interferometer a thickness of 100 A is permissible, but the single-crystal structure must be perfect enough so that the diffraction pattern will consist of sharply defined spots when an electron beam is passed through the film. For the scattering experiments, films of a single chemical species were desired, and to make negligible the possibility of plural and multiple inelastic scattering, a thickness of less than 100 A was indicated. It appeared that single-crystal metal films of this degree of thickness would be satisfactory for both applications, and that such films might be grown successfully on a substrate that would orient the growing deposit. Knowledge gained from an analysis of the experimental data on the formation of such films, along with new additions to the theory, are presented in this review of the process of epitaxy.

2. Analysis of Existing Data

The term epitaxy has been coined by Royer [4] to designate the oriented overgrowth of one crystalline substance on another. Although Royer's work was concerned entirely with growing a salt from its aqueous solution onto a single crystal of a different salt, epitaxy may also result when the deposit material is condensed from the vapor phase onto the substrate, when the deposit material is electrodeposited onto the substrate, or when the deposit material is produced as a product of a chemical reaction that takes place on the surface of the substrate.

Results of past experimental work have indicated two general conditions that are necessary for the formation of oriented single-crystal metal films. These are (1) a certain amount of fit between the lattice of the substrate and that of the deposit material, and (2) a characteristic substrate temperature (epitaxial temperature) associated with every metal-substrate pair below which perfect orientation is not obtained.

For use in the interferometer and in the scattering experiments, the film must be removed from the substrate. Rock salt, which is easily obtained in single-crystal form, and which can be removed from the metal films by solution in water, is a satisfactory substrate from this standpoint. By using substrate temperatures that are tabulated in the literature for about six metal–rock salt pairs, very thin films of a number of these metals have been condensed from the vapor phase by J. A. Suddeth and T. F. McCraw of the Electron Physics Section of the National Bureau of Standards. Not all of these films have desirable properties. The gold films are very nicely oriented if the rock salt cleavage face is maintained at 400° C during the deposition of the metal. They are transparent; blue light is transmitted. They have, however, the undesirable property that they collapse under their own weight at this extraordinary thickness because of the low interatomic forces of attraction between the gold atoms. Figure 1 is the electron diffraction pattern of one of the few gold films of this order of thickness that were successfully mounted by Suddeth and McCraw.

The electron diffraction patterns of silver films show that these films are well oriented if the cleavage face of the rock salt is maintained at 150° C during the deposition of the metal. Silver films of this order of thickness are sturdy but are full of small holes. See, for example, the electron micrograph shown in figure 2, where the angular structure of the holes (angles of 45 and 90 degrees) clearly indicates the cubic lattice type for silver. Johnson [5] has followed the structural changes of thin oriented silver films that were condensed in a vacuum on rock-salt cleavage faces as a function of temperature. For films of 1000 to 2000 A he found that rectangular

[1] Work sponsored by the Office of Naval Research.
[2] Figures in brackets indicate the literature references at the end of this paper.

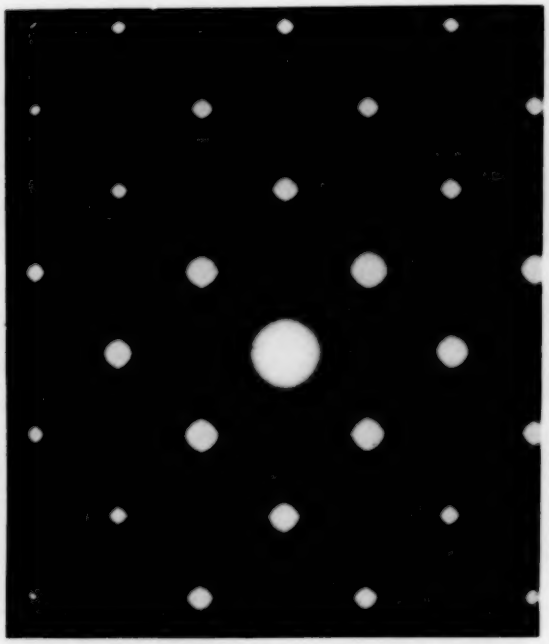

FIGURE 1. *Electron diffraction pattern of a thin gold film evaporated onto the cleavage face of rock salt.*

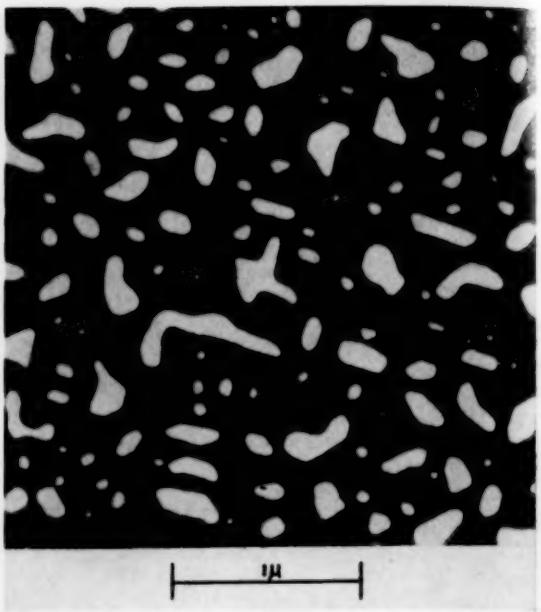

FIGURE 2. *Electron micrograph of a thin silver film evaporated onto the cleavage face of rock salt.*

holes appeared in the films. Their edges were alined with the [110] direction of the rock salt. He attributed the effects to the readjustment of the oriented surface under the influence of surface-tension forces.

To extend experimental investigation to a variety of metals for which data are not available in the literature, it became desirable to be able to predict reasonably well not only the best substrate for maximum orientation but also the characteristic epitaxial temperature to be expected for the pair. Generalizations comparable to the rules of Royer [4] for epitaxial deposits from solution do not exist in the case of condensation of metals from their vapors. The following approach to the problem is an analysis of existing data to find a basis on which such predictions can be made.

2.1. Criterion of Best Fit

In regard to the first condition, it appears that in the case of metal films deposited from the vapor in a vacuum too much importance has been given to the degree of lattice fit between the deposit and substrate. This overemphasis of lattice fit seems to have grown out of the extensive work of Royer [4], Sloat and Menzies [6], and others on the epitaxy of salts deposited on various substrates from solution under which condition the degree of fit was apparently rather critical and the dielectric properties of the solvent played a role. It was further entrenched by the mathematical treatment of Frank and van der Merwe [7, 8]. Recently, however, Schulz [9] has concluded, even for deposition from aqueous solution, that close atomic matching at the interface is not required for an oriented overgrowth. He obtained oriented films by deposition from the vapor phase. He then dissolved these films by condensing water vapor on them. Finally, he regenerated the oriented film by evaporation of the water. In evaluating his conclusion one must bear in mind that in the method that Schulz used the ions from an oriented deposit were just barely brought into solution by condensation of water vapor onto the film without a mechanical action so that the ions of the first oriented monolayer on being brought into solution probably never strayed far from the site on the substrate that they had occupied. In this way the oriented nucleus in solution continued to exist. Pashley [10] has oriented silver chloride on mica from the vapor phase with a misfit of −24 percent in spite of the fact that Royer [4] found the misfit limit of rock-salt lattice-type salts on mica from solution to be about 15 percent. On the basis of this evidence Pashley has suggested that the method of growth of oriented deposits is an important factor to be taken into account.

While epitaxy that results from crystallizing the deposit material from solution may take place under fairly critical conditions of fit, results of experimental work on the deposition of metal films from their

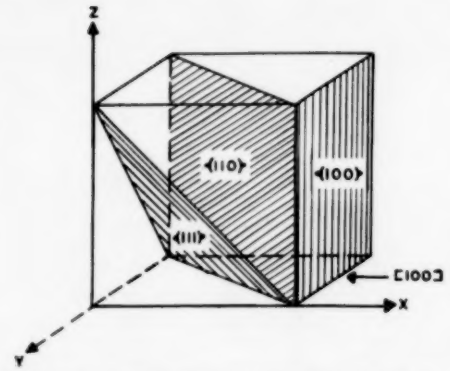

FIGURE 3. *The {100}, {110}, and {111} faces of a cubic crystal structure.*

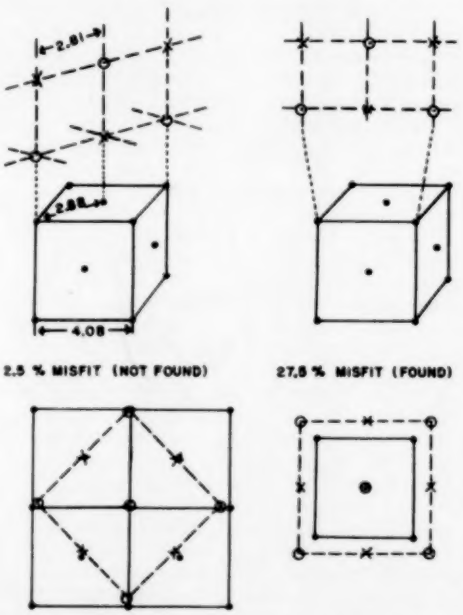

FIGURE 4. *Silver on the {100} face of rock salt at 150° C.*

●, Silver; X, sodium ion; ○, chloride ion

vapors have shown that there is a wide tolerance in fit in many cases in which epitaxy has been found, and that the depositing vapor often chooses to condense in an orientation with respect to the substrate that from the standpoint of fit would be undesirable. To illustrate the point the orientation of silver deposited on the {100}, {110}, and {111} faces of rock salt will be reviewed (see fig. 3).

Brück [11] in depositing silver from the vapor in a vacuum onto the cube face of rock salt at the characteristic epitaxial temperature for perfect orientation on that face found that the silver single crystal

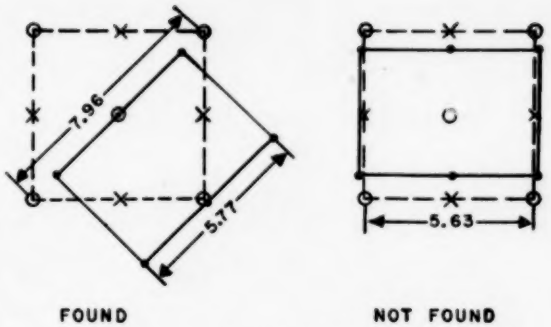

FIGURE 5. *Silver on the {100} face of rock salt at 450° C.*

●, Silver; X, sodium ion; ○, chloride ion

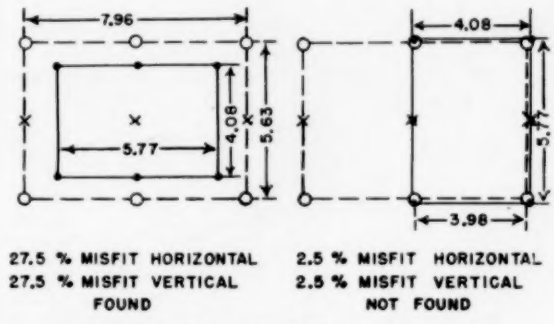

FIGURE 6. *Silver on the {110} face of rock salt.*

●, Silver; ○, chloride ion; X, sodium ion

grows with its {100} face resting on the substrate and lying parallel to the cube edge of the rock salt with 27.5-percent misfit in spite of the fact that it would have a better fit (2.5-percent misfit) if it grew turned at 45 degrees to the sodium chloride cell (see fig. 4). Gharpurey [12] deposited silver on the cleavage {100} face of rock salt at an elevated temperature (450° C) and found that at this temperature the silver deposits with its {110} face resting on the {100} face of the rock salt. It was inclined at 45 degrees to the rock-salt cube with misfits of 2.5 percent and 27.5 percent along the two directions of the rectangular parallelogram of the {110} face of the silver cube. From figure 5 one sees that an orientation parallel to the rock salt cube would have given the same percent misfits. It is, however, not reported.

Brú and Gharpurey [13, 14] found that when silver is deposited from its vapor onto a {110} face of rock salt, the {110} face of the silver rests on and is parallel to the {110} face of rock salt. The misfit is 27.5 percent in each direction considering the chlorine lattice. If the silver were rotated through 90 degrees, the fit would be nearly perfect as is clear from figure 6, but this orientation is not reported. Similarly, on depositing silver from the vapor onto the {111} face of rock salt, they found that the {111}

face of silver was resting on the {111} face of sodium chloride (see fig. 7). The misfit on the {111} plane is 27.5 percent for the silver lattice.

Brück [11], who investigated the orientation of nine different metals on the cleavage face of rock salt, observed that whereas the cubic face-centered metals grew with their {100} face resting on the {100} face of the rock salt and oriented in the same way as the rock salt (i. e., with their cube edge parallel to the rock-salt cube), the cubic body-centered metals grew on the contact surface turned at 45 degrees to the rock-salt cube. In the case of the cubic body-centered metals, then, the orientation giving best fit is observed.

From this evidence it appears that best fit is not the only criterion that determines whether or not epitaxy will occur in metals deposited from their vapors, or what orientation will be observed if it does occur. There appears to be some other factor involved that is more important than fit. An interpretation of the orientations that have been observed will be discussed at length in section 5, and the experimental observations reviewed here will be reexamined in the light of that discussion.

Brück found experimentally (by trial and error) the epitaxial temperature to produce single-crystal films of Cu, Pd, Au, Ag, Ni, and Al on the {100} face of rock salt. He tabulated these temperatures along with the lattice constant for each of the metals as is reproduced in table 1 without advancing an explanation for the differences in temperature. The most marked of these differences is that between silver and gold for which the lattice constants are nearly identical. Each of the metals listed has the cubic face-centered lattice. Three metals, Fe, Co, and Cr, for which Brück was unable to reach the epitaxial temperature due to the fact that the rock-salt substrate sublimes at 540° C are not included in the table.

TABLE 1

Deposit metal	Lattice constant	Epitaxial temperature
	A	°*C*
Ni	3.52	370
Cu	3.60	300
Pd	3.86	250
Al	4.04	440
Au	4.07	400
Ag	4.08	150

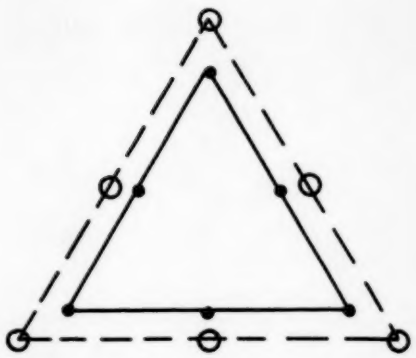

FIGURE 7. *Silver on the {111} face of rock salt.*
●, Silver; ○, chloride ion

2.2. Epitaxial Temperature

As has already been stated, one of the conditions for laying down an oriented deposit of a metal from the vapor phase onto a substrate is that there exists a minimum substrate temperature, the epitaxial temperature, below which perfect orientation is not observed.

Uyeda [15] has extended the investigation to a variety of substrates.

The data of Brück and those of Uyeda for the deposition of silver from tabulations given by Uyeda are listed directly in table 2.

From this set of data it appears that the epitaxial temperature for orienting a given face parallel to the substrate (for example, the {100} face of silver parallel to the {100} face of galena, iron pyrite, and rock salt) is not very different for different substrates. The temperature for laying the {100}, {110}, or the {111} face of a given material parallel to a given substrate, however, is different. Brück observed that, in general, the most densely populated plane tends to go parallel to the substrate at a relatively low temperature; then comes the next most densely populated at a higher temperature, and so on. The most densely populated plane for cubic face-centered metals is the {111} plane; for cubic body-centered metals, it is the {110} plane.

The question of what determines the epitaxial temperature for a given element remains as yet

TABLE 2

Substrate	Crystal type	Orientation	Epitaxial temperature
			°*C*
Molybdenite	C7	$(0001)_{MoS_2}\|\|(111)_{Ag}$; $[10\bar{1}0]_{MoS_2}\|\|[110]_{Ag}$	<20
Zincblende	B3, B4	$(110)_{ZnS}\|\|(110)_{Ag}$; $[110]_{ZnS}\|\|[110]_{Ag}$	80~100
Galena	B1	$(100)_{PbS}\|\|(100)_{Ag}$; $[100]_{PbS}\|\|[100]_{Ag}$	100~150
Iron pyrite	C2	$(100)_{FeS_2}\|\|(100)_{Ag}$; $[100]_{FeS_2}\|\|[100]_{Ag}$	~150
Rock salt	B1	$(100)_{NaCl}\|\|(100)_{Ag}$; $[100]_{NaCl}\|\|[100]_{Ag}$	150

unanswered. Smollett and Blackman [16] have stated that is is hard to understand the temperature dependence of orientation in terms of critical misfit alone. Although it has been suggested that the epitaxial temperature may serve the purpose of expanding the lattice, such expansion can hardly play an important role since Eppelsheimer and Penman [17] have shown that the copper lattice constant increases only slightly over 1 percent between 18° and 770° C, which is negligible in comparison with observed misfit in epitaxy. Neuhaus [18] has associated the epitaxial temperature with the lattice energy of the deposit and has concluded that the oriented deposits are not the result of adsorption (partial energy exchange) but are the result of a true two-dimensional reaction (complete energy exchange).

3. Process of Epitaxy

Since the first experimental work on oriented deposits was done, attempts have been made to explain the observed orientations. At the end of a very extensive experimental investigation, Royer [4] summarized his observations in three empirical rules. These were (a) that within a certain tolerance the geometric distribution of lattice sites in the plane of the substrate and in that of the deposit should be identical or the distribution in one should be a simple multiple of that in the other, (b) that the deposit ions, which take positions that substrate ions would have taken had the substrate continued to grow, should have the same polarity that the substrate ions would have had, and (c) that the type of binding between elements of the deposit should be identical with that between elements of the substrate. He wrote the third rule with hesitancy since he was not certain that the evidence that was available really proved it. Since then, Sloat and Menzies [6] found orientation of ammonium bromide and of cesium chloride deposited from aqueous solution onto a single crystal of silver. They interpreted the observed orientation by assigning positive charges of not necessarily effectively unit value to the atoms of the silver crystal. They assumed that these charges alined the negative ions that form lattice planes in the salts. Johnson [19] found epitaxy between sodium chloride deposited from solution onto oriented films of silver, which had the {100} and the {111} planes parallel to the surface, and on polycrystalline silver sheet. He concluded that Royer's third rule should be modified or abandoned.

One of the earliest of the theoretical treatments of epitaxy was that of Dixit [20]. He considers that the first layer of adsorbed atoms in random lateral motion on an unattractive (amorphous) solid surface may be represented as a two-dimensional gas obeying essentially the ideal gas equation

$$\pi A = RT. \tag{1}$$

Here π is the two-dimensional pressure, A is the surface area per adsorbed molecule, R is the gas constant, and T is the temperature of the surface. The two-dimensional pressure π is given by

$$\pi = \gamma_0 - \gamma, \tag{2}$$

where γ_0 is a constant and γ is the surface tension. By substituting the value of γ given by the Ramsay-Shields equation, and by substituting successively the area per molecule in the {100}, {110}, and {111} plane, he succeeded in showing that the temperature at which any of these orientations appear is directly proportional to the area A in this plane. By means of additional assumptions, he was able to calculate appearance temperatures for these planes that compared very favorably with those observed. Although his treatment applies to an unattractive substrate, the difference in the appearance temperatures of these planes should be the same on any substrate if the assumption of a two-dimensional gas can be applied to the diffusion process on the substrate surface.

Brück was aware of Dixit's work at the time that he discussed his own experimental results, but he regarded the treatment as inadequate although it was in line with his own observation that the densest plane of the deposit tends to go parallel to the substrate first. He noticed the seeming preference for misfit in the orientations of nine metals on the cleavage face of rock salt that he observed experimentally, and attempted to explain the results by saying that the orientation appeared to be such as to make the sum of the distances of the substrate ions of one species from the deposit atoms a minimum in a cell. He says that the arrangement takes place in such a manner that for the range of an elementary cell the sum of the distances of the atoms of the uppermost layer of the substrate from those of the deposit lattice will be as small as possible. Brück's rule has been supported by Uyeda, as we shall see in a moment, but it has been criticized both by Brú and Gharpurey [14] and by Thomson and Cochrane [21].

Thomson and Cochrane point out that what would be a favorable orientation in the first cell if Brück's rule were obeyed would be an unfavorable one later on. This is, of course, true, and in this respect Brück's rule is inadequate although it points in the right direction. If Brück's rule is extended, however, it can be used to define the most favorable orientation in terms of attractive bonding energy to the substrate for the largest area of monolayer deposit lattice that is capable of forming over the substrate in question. This will be discussed in more detail in section 5. Brú and Gharpurey dismiss Brück's explanation as unconvincing in that among ions of the same polarity the distances have rather to be large than small [13]. However, it does not appear to be necessary to minimize a distance-sum over ions of the same polarity. It is, for example, not necessary to conceive of the {111} face of rock salt as being made up of sodium ions. To be absolutely realistic, the best {111} face that one could cut experimentally would undoubtedly house both ionic species. It appears that the comment of Brú and Gharpurey must have been directed

toward the fact that Brück, in considering the cleavage face of rock salt, imagined the sodium ions deleted from the lattice, so that a simple face-centered cubic structure of chloride ions remained. It was in a cell of this modified lattice that his rule applied. An extension of Brück's rule to include antibonding as well as bonding energy will be discussed at length in section 5.

Thomson [22], and Thomson and Cochrane [21] thought that steps and cracks on the surface are often a determining factor in producing epitaxy. Wilman [23] rejects the explanation of Thomson and Cochrane. He states that rotational-slip results show that the existence of angles or edges of cleavage steps, whether these are macroscopic or only monatomic in height is quite unnecessary for epitaxial orientation to occur, but that if such steps or cracks are present, they may often determine the location or habit of the deposit crystals.

Uyeda [15] regards the cracks and steps as contributing only a disturbance to the ideal case. He regards the preferred orientation as being induced by the periodic field of surface force, and has advanced the following two assumptions in terms of which to discuss the oriented growth of crystals: (1) The configuration of atoms in a crystal bud formed on a base is almost the same as that formed in a free space, and (2) the interaction between the base and the crystal decides the orientation of crystals. He concludes that the orientation should be fixed while several atoms are gathering around a nucleus (compare Brück's orientation in an elementary cell) and states that this point of view supports Brück's rule of orientation and explains a few phenomena concerning orientation qualitatively. He has, however, not made clear what he means by interaction between the base and crystal. Kainuma [24] has found evidence in favor of Uyeda's conclusion.

Kubo and Miyake [25] suppose that the vaporized metal atoms, after wandering on the surface, finally arrive at the foot of a nucleus and it is from here the crystal growth commences.

The mathematical treatment of Frank and van der Merwe [7, 8] has already been referred to. Their theory is based on the properties of a one-dimensional dislocation model. It consists of a row of identical balls connected by identical springs having force constant μ. The balls are acted on by a force that varies periodically with the position on the substrate. They let a be the wavelength of the potential due to the substrate and $W/2$ be its amplitude. The potential energy is represented by the first harmonic term (amplitude $W/2$) in a Fourier series. A difference may exist between the natural spacing b of the balls and the wavelength a of the substrate field. From the theory for the model it turns out that when the natural spacing b of the balls differs from the wavelength a of the substrate field, the lowest energy state of the system is one in which there are no dislocations (small regions in which the balls would be spaced more closely or less closely than normal) up to a certain critical value of the misfit $1/P_0$, where

$$1/P_0=(b/a-1)_{\text{critical}}=2/\pi l_0 \tag{3}$$

and

$$l_0=\left(\frac{\mu a^2}{2W}\right)^{1/2}. \tag{4}$$

By using Lennard-Jones forces, and by assuming interactions with other deposit atoms and with substrate atoms are similar, they find the value of l_0 to be about 7 and hence the critical value of the misfit to be about 9 percent. Below this critical misfit the monolayer in its lowest energy state is deformed into exact fit with the substrate. Above the critical misfit it is only slightly deformed on the average, but it contains many dislocations. Up to the larger critical misfit of 14 percent the monolayer can be deposited metastably in exact fit on the substrate at temperatures that are sufficiently low. Completely oriented overgrowth is to be expected only if the first monolayer can form over the entire surface under subcritical conditions.

Smollett and Blackman [16] have investigated the stability of a strained two-dimensional layer. They evaluated the lattice constants of a two-dimensional alkali halide layer strained to fit on an alkali halide crystal substrate. For a series of cases in which oriented overgrowth is found experimentally they found instability in the initial deposited layer if it is strained to fit the substrate. They conclude that experimentally observed orientations cannot result from an initially uniformly strained layer such as is indicated by the Frank-van der Merwe theory. They further conclude that the initial lattice must be distorted in the sense that displacements occur, but that the lattice constant of the deposit material remains more or less unchanged.

In analyzing the forces that operate in epitaxy from solution, Sloat and Menzies [6] state that the most important are the mutual attraction between the solute ions and the substrate ions, and the mutual attraction between oppositely charged solute ions. Following Lennard-Jones and Dent [26] they further analyze the first force mentioned into four parts: (1) Electrostatic attraction of the ionic charge to the valency charges of the substrate; (2) the force of attraction between substrate ions and the dipole which results from the polarization of the attracted ion by the surface field; (3) the attractive force resulting from the polarization of the surface ions by the attracted ion; and (4) the van der Waals attraction between neutral atoms. The van der Waals attraction is operative at longer distances, but at short range the electrostatic force is much greater in magnitude. They conclude that it is first the van der Waals force which is operative; then the electrostatic. They ascribe the final capture of an ion, and the fixing of the ion into the space lattice of the substrate, to the electrostatic forces.

It appears that a similar mechanism must operate in the epitaxy of metals deposited from their vapors, although Rhodin [27, 28] has found rather surprising agreement on the assumption of van der Waals

forces. However, his criterion of the completion of the deposition of a monolayer was conductivity across the film. It is hard to believe that a continous monolayer of metal is ever produced because of the strong cohesive forces of the deposit. Rather it would seem that aggregates of a number of layers in depth eventually grow together. One can then hardly talk about the binding force of the first layer from a conductivity measurement.

4. A Model for Metal–Rock Salt Pairs

In order to establish a basis on which prediction of metal-substrate pairs for vapor-phase deposition and of their epitaxial temperatures can be made, it is necessary to know the process involved. If this process can be understood for one substrate, it can be generalized to others. With this view in mind the special case of the epitaxy of metals on a rock-salt substrate has been considered [28a] because a larger amount of experimental data exists for this substrate.

We shall start with the following assumptions: (1) Orientation must result from the interplay of forces, and (2) the preferred orientation must be the configuration of lowest energy. If we assume that the first layer of deposit-metal exists as atoms, the forces, in addition to van der Waals forces, that are involved are electrostatic forces acting between the substrate ions and the dipoles induced by the substrate ions in the neutral metal atoms. If, on the other hand, we consider that the atoms of the first layer of deposit-metal become ionized, they are electrostatic forces acting between the ions of the substrate and deposit-metal ions on the surface. When neutral metal atoms arrive on the surface of the substrate, forces of the first kind are operative. If, after arrival on the surface, the deposit-metal atoms permanently lose one or more electrons, forces of the second kind exist. A true two-dimensional reaction can then occur between the deposit-metal ions and the anions of the substrate so that at the interface the three layers—substrate salt, deposit-metal salt monolayer, deposit-metal—exist. The deposit-metal salt that forms as a result of this reaction must be sufficiently mobile to diffuse on the surface possibly in the form of symmetrical units, or lattice cell fragments, such as discussed by Zener [29] until larger static aggregates are formed. It would appear that the epitaxial temperature should be related not only to the mobility of this salt, which in the form of larger aggregates is eventually firmly anchored to the substrate by true electrovalent bonds, but that the epitaxial temperature is also a function of the ionization potential of the respective metals. Thermal ionization at the expitaxial temperature is, of course, out of the question because of the vanishingly small value of the Boltzmann factor. But the combined effect of thermal energy and the availability of electron traps on the surface could be sufficient as we shall see in a moment. If one plots the first ionization potential of each of these metals against the epitaxial temperature, there is no relation between them. If, on the other hand, one plots the ionization potential of the most commonly encountered ionization state, the trend is along a line (see fig. 8, and reference [32]). For these metals the only valence states used that are higher than the lowest ionization state possible for the atom in question under usual conditions are Cu II, Au III, and Ni III. We shall return shortly to a discussion of the reasonableness of assuming that these will be the most stable valence states to be expected for these metals, and we shall consider first whether or not the process that has been advanced is energetically feasible.

In considering what could constitute trapping sites for the electrons, one might at first sight fix upon the sodium ions. That this is not a logical choice, however, is evident from the fact that if rock salt is heated in sodium-metal vapor, F-centers are formed in preference to the alternative that the lattice should house sodium metal interstitially. The F-centers (Farbzentren, or color centers) are vacant chloride ion sites, which due to their effective positive charge, have trapped an electron whose orbit then extends over the six surrounding sodium metal ions. They may be regarded as a vacant negative ion site at which one of the nearest neighbor positive ions has gained the electron and become a neutral metal atom. If we write M for the deposit-metal, S for a vacant negative ion site, and F for a color center, then we are concerned with the energy for the two-step process,

$$\text{Step (1):} \quad M^0 - ne \rightarrow M^{n+} \tag{5}$$

$$\text{Step (2):} \quad nS + ne \rightarrow nF. \tag{6}$$

The second step in sodium chloride involves about 2.7 ev. This is the peak of the characteristic absorption band of the F-centers [30]. The metal atom, let us consider it to be silver, is first adsorbed from the vapor onto the cleavage face of the rock salt. After a certain period of time, during which it is adsorbed on the surface, loss of an electron takes place. Since the silver atom is on the surface of the sodium chloride, the energy required is less than the gas phase ionization potential of silver. That is, the ionization potential is reduced because of the dielectric constant of the salt. Because a thermal rather than an optical process is involved, the static dielectric constant K is in question rather than K optical arising from the distortion of the electron clouds. The whole value of K would be effective only if the process took place below the surface of the crystal, that is, if the adsorbed atom were completely dissolved in the rock salt. If the cleavage plane were atomically smooth, one might reasonably assume that the dielectric constant should be reduced by somewhat less than half. Actually the factor 0.44 has been used as this gives the lowest positive value for ΔE in the case of silver for which the data are known most accurately. In addition to the energy available from eq (6) there is also available the energy of placing a Ag^+ ion over a Cl^- ion on the cleavage plane. If the cleavage plane were atomically smooth, this energy would be given by [31]

$$E = \frac{0.066\, e^2}{r}, \tag{7}$$

where ϵ is the electronic charge, and where r is the Ag^+–Cl^- distance of separation. The first ionization potential of silver [32] is 7.574 ev, the dielectric constant of rock salt [30] is 5.62, and if we take the Ag^+–Cl^- distance to be the sum of the ionic radii (Goldschmidt radii are $Ag^+ = 1.13$ A, $Cl^- = 1.81$ A), then

$$\Delta E = \frac{7.574}{0.44K} - 2.7 - \frac{0.066\,\epsilon^2}{r} = 0.04 \text{ ev.} \qquad (8)$$

In the case where a doubly charged ion is produced, as in Pd^{++}, an additional energy is involved due to the polarization of the surface ions by the double positive charge. Wright [33] gives the potential when an electron is added at a positive ion site in sodium chloride as −2.5 ev. It would appear that on the surface, although the Pd^{++} ion has only one nearest neighbor Cl^- ion, it sees all of the ions in the plane on which it is standing, so that the polarization energy is a large fraction of that which results when an electron is given to a positive ion in the interior of the salt. A reasonable guess might be 0.7 of this value or 1.75 ev. In the case of forming a doubly charged ion, two F-centers are involved and the energy for placing the ion on a Cl^- ion in the plane is doubled. Because the second ionization potential of palladium is 19.42 ev [34], the energy for producing the Pd^{++} ion is given by

$$\Delta E = \frac{19.42}{0.44K} - 2(2.7) - 2(0.066\,\epsilon^2/r) - 0.7(2.5) = 0.01. \qquad (9)$$

There is little point in estimating the energy change for a triply charged ion such as Au^{+++} because of the uncertainty both in the value of the ionization potential and in the polarization energy.

This value of ΔE, however, does not determine the equilibrium constant. It is really the thermodynamic ΔE_0^0 or energy change for the reaction at the absolute zero of temperature. The equilibrium constant K is given by

$$-\ln K = \Delta F^0/RT, \qquad (10)$$

where R is the gas constant, T is the absolute temperature, and ΔF^0 is the standard free-energy change.

$$\Delta F^0 = \Delta E_0^0 + \Delta\left(\frac{F^0 - E_0^0}{T}\right) T. \qquad (11)$$

For the ionization process postulated, the second term on the right for silver is

$$T\left[\left(\frac{F^0-E_0^0}{T}\right)_{Ag^+} + \left(\frac{F^0-E_0^0}{T}\right)_{F\text{-center}} - \left(\frac{F^0-E_0^0}{T}\right)_{Ag^0} - \left(\frac{F^0-E_0^0}{T}\right)_{\text{vacant-site}}\right]$$

This information is not available and consequently the equilibrium constant cannot be calculated.

If it is conceded that ionization of the deposit atoms does occur on the surface of the substrate, and the evaluation of the energies involved appears to indicate that such a process is feasible, then it must also be conceded that the epitaxial deposits consist of the layers: rock salt, deposit-metal salt, deposit-metal, of which the deposit-metal salt layer is a monolayer. If this is indeed true, we must expect in every case in which epitaxy is observed that there is epitaxy between rock salt and the deposit-metal salt as well as epitaxy between the deposit-metal salt and the deposit-metal, and we must consider the lattice fit between the salt lattices, and that between the deposit-metal salt lattice and the deposit-metal lattice rather than the fit between the metal lattice and the rock-salt lattice as has been done so far.

Following this point of view, when the silver metal itself deposits, it does so on an oriented surface of silver ions. Here, three cell lengths of the silver chloride lattice (16.62 A) are closely equivalent to four cell lengths of bulk silver (16.32 A). The ease with which the films can be floated from the salt cakes seems to indicate that they are not bonded to the substrate cell by cell such as an additional layer of sodium chloride would be. Rather, there must literally be tunnels through which the water can flow under the film.

Some very interesting experimental work has been done recently by Johnson [35] in regard to epitaxy of salts on metal surfaces. He found epitaxy between rock salt and the following metals: Cu, Ag, Au, Zn, Pb, Sb, and Bi, when he evaporated a salt solution on the metal. Two metals known to exhibit epitaxy when deposited from the vapor onto a rock-salt cleavage plane were negative to Johnson's test. These were nickel and aluminum. While one might ascribe the negative result in the case of nickel to the large misfit between the nickel lattice and the rock salt lattice, one could hardly advance this argument in the case of aluminum where the lattice constant is very close to that of silver and gold. The answer may then possibly be that it is not necessary that there should be epitaxy between rock salt and the metal itself, but that it is only necessary that there should be epitaxy between rock salt and the metal salt; and again, epitaxy between the metal salt and the metal as has just been indicated. Johnson explained his results by assuming the presence of an oriented oxide layer on the metal. Where the oxide layer was known to be amorphous, orientation did not occur. In the case of silver and gold no oxide layer was considered to be present. In this connection Johnson states that it is possible then that the oriented growth of sodium chloride on silver did involve an intermediate thin layer of silver chloride. Silver produced a much better orientation of the sodium chloride than did the gold. In regard to this, Johnson states that the difference in behavior of silver and gold must be attributed to differences in their chemical activity. Brück [11] also found that silver was a more efficient substrate material than gold as is evident from the epitaxial temperatures that he found for deposit-substrate pairs given in table 3.

TABLE 3

Substrate	Deposit	
	Ag	Au
	°C	°C
NaCl	150	400
Ag	100	300
Au	250	----

Sloat and Menzies [6] have shown that sodium chloride can orient silver chloride and Wilman [36] has shown that silver can orient silver chloride. Consequently, orientation is possible through the three layers: NaCl, AgCl, and Ag. In the list of positive and negative cases of epitaxy given by van der Merwe [8] it is indicated that copper does not orient cuprous chloride. It remains to be shown that copper does orient cupric chloride since according to the plot of epitaxial temperature against ionization potential shown in figure 8 one is led to believe that the divalent salt forms. Only by obtaining similar data for the other metals for which the epitaxial temperature on rock salt is known, can it be shown that the valence state indicated by the epitaxial temperature is the only ionization state for which the metal salt will orient both the substrate salt and the metal deposit.

Of the metals so far considered (data of Brück [11]) the only valence states used that are higher than the lowest state possible for the elements in question under usual conditions were Cu II, Au III, and Ni III. In determining whether or not chlorides in these valence states would be stable at the epitaxial temperatures required, one must consider both the stabilizing effect of the sodium chloride and the fact that a monolayer is in question. Morris [37], for example, has found that auric chloride solution containing sodium chloride is stable to 450° C but auric chloride by itself decomposes at a much lower temperature. What effect stabilization due to the sodium chloride may have in determining the valence state that should be observed can only be surmised. The chemistry of thin layers, let alone monolayers, has been investigated very little. In regard to the use of the higher oxidation state for gold, it is noteworthy that aurous chloride undergoes dismutation with ease to produce auric chloride and gold.

The limitations of the graph shown in figure 8 must be clearly borne in mind. Each of the six metals on it has the cubic face-centered lattice type. The epitaxial temperature for each is for deposition of the {100} plane parallel to the {100} plane of rock salt. It is to be noted that for face-centered cubic lattices, the orientation with the {111} plane parallel to the substrate should occur at a lower temperature than that for orientation of the {100} plane parallel to the substrate. However, the orientation with the {111} plane parallel to the substrate in many cases never appears. Brück, for example, observed that this was true for copper, palladium, nickel, and cobalt. He found that the {111} orientation parallel to the substrate did appear

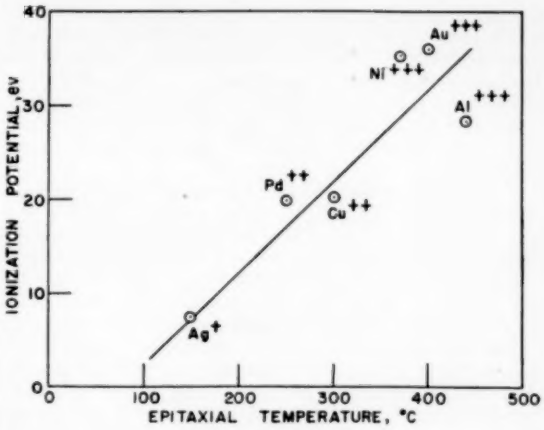

FIGURE 8.

for gold and aluminum and that there was some indication of its appearance for silver. The fact that it does appear in aluminum, and must be forced out by increase in temperature, may account for the poorer fit of aluminum on the graph. If the epitaxial temperature for laying down the {111} plane of aluminum parallel to the substrate (which may be taken to be about 350° C since above this value the {100} plane starts to deposit) were used for aluminum, this metal would have a much better fit on the graph. This, too, would be in line with the assumption that the metal must be ionized to be oriented at all. Gold would also be displaced to the left on this reasoning.

In the discussion of Dixit's investigation in section 3, it was pointed out that although his treatment applies to an unattractive substrate, the differences in his appearance temperatures should be correct for any substrate if the assumption of a two-dimensional gas can be applied to the diffusion process on the substrate surface. For silver, Dixit finds the {111} appearance temperature to be 659° C and that of the {100}[3] face 801° C. The difference is 142 degrees. Brück reports that silver can be oriented fairly well at room temperature. At 150° C he obtained single crystal orientation with the {100} face parallel to the substrate, but sometimes, in the foils nearly oriented to a single crystal, there was a light superimposition with the {111} structure. Assuming that the first observed orientation was with the {111} plane parallel to the rock salt, the temperature range between the two orientations is about 130 degrees. For aluminum on an unattractive substrate Dixit found for the {111} appearance temperature 264° C, for the {100} appearance temperature 346° C, and for the {110} appearance temperature 602° C. The difference between the appearance temperatures for the {100} and {111} planes is 82 degrees. In regard to aluminum Brück states that above 350° C the single crystal drives out gradually the {111} structure until

[3] Dixit, who was considering interplanar spacings, used the notations [200] and [222] instead of {100} and {110}.

it only exists at 440° C. The aluminum film at 440° C has the {100} plane parallel to the rock-salt cleavage plane. This is a temperature range of 90 degrees, which again compares favorably with the difference in Dixit's calculated values. This agreement appears to indicate that the same assumption of a two-dimensional gas may be applied to attractive substrates.

There appear then to be two temperature dependent processes: (1) the ionization process and (2) the process by which the {111}, {100}, and {110} planes orient parallel to the substrate. The second process is a complicating factor in the first. The answer to the question as to why orientation with the {111} plane parallel to the substrate does not make its appearance for every metal is not clear. It must be energetically forbidden. The answer may be simply that the temperature required to produce ions that can be oriented at all in sufficient number is already in excess of that at which the {111} plane would orient parallel to the substrate if ions had been available. Then the next most densely populated plane, the {100} plane for the face-centered cubic structure, would show up.

5. Retrospect

In this section we return to the experimental results that have been cited for the deposition of metals from their vapors onto a rock salt substrate to find if the ionization process that has been advanced can explain the orientations which have been found. At its epitaxial temperature, Brück found that silver deposits on the cleavage face of rock salt in the way indicated in figure 4, although, if there were a relative rotation of 45 degrees, the orientation designated as not found in figure 4, the misfit would be very much less. It is interesting to observe from the two-dimensional sketches that the misfit orientation that is actually found places a chloride ion in closest proximity to a silver atom (or silver ion, in terms of the ionization process discussed) and leaves the sodium ions with no silver close by. If the silver cube diagonal did lie along the rock-salt cube edge (the orientation involving best fit, but which is not found) a sodium ion would make a close approach to a silver atom. If the silver atom is really a silver ion, this would involve repulsion. If silver ions, rather than silver atoms, are involved, the orientation observed is in accord with Royer's second rule, which says essentially that you will not get epitaxy if ions of the wrong polarity have to be brought together. This rule is for the orientation of charged ions depositing out of solution. The fact that it indicates as impossible the orientation that is actually not found, but which one would expect to find on the basis of best fit, lends support to the idea that charged ions are involved. The orientation that is found appears to try to establish as closely as possible a monolayer lattice of silver chloride using the anions of the substrate. The lattice constant of silver chloride (a=5.545 A) is nearly the same as the lattice constant of sodium chloride (a=5.628 A). A close approach to this hypothesis was made by Drabble [38] who concluded that the epitaxy is such that the nearest possible coordination fit between the silver chloride (sodium chloride type structure) and silver structures occurs across the interface. The conclusion of Drabble is cited by Brú and Gharpurey [13, 14] in regard to their own experimental results.

In the deposition of silver on the {100} face of rock salt at 450° C one sees again in the orientation not observed that sodium ions would be brought close to silver ions and the situation is the same as that just discussed (see fig. 5). It was pointed out in regard to the deposition of silver on the {110} face of rock salt that if the silver were rotated through 90 degrees, the fit would be nearly perfect, but that this orientation is not found. The orientation of best fit which is not found experimentally would again involve a close approach of the silver ions to the sodium ions (see fig. 6).

As was also pointed out previously, Brück found that whereas the cubic face-centered metals grew with the {100} plane resting on the {100} of rock salt and oriented with their cube edges parallel to the rock-salt cube, the cubic body-centered metals grew on the contact surface turned at 45 degrees to the rock salt cube. From figure 4 it is clear that in the case of a cubic body-centered metal, there would be no interference from repulsion of sodium ions in depositing such a metal with its cube diagonal along the sodium chloride cube edge because there would be no deposit-metal ions in these positions.

We see that although Brück's rule points in the right direction, it needs to be extended. If we consider that electrostatic forces are involved, we must say that the arrangement is such as to give maximum bonding to the substrate for the largest aggregate that is capable of forming. One must then sum the attractive bonding energy, and the repulsive antibonding energy between each silver ion and its four nearest neighbor substrate ions of each species over the area that is defined by the points where exact fit is obtained between the substrate salt lattice and the deposit-metal salt lattice. If the sum of the attractive energies (−) added to the sum of the repulsive energies (+) is negative, the arrangement is stable, but if it is positive, the arrangement in question should not exist. It should be possible to explain all of the orientations that have been found by determining these sums graphically both for these orientations and for other possible orientations which are not observed. It should also be possible to explain the laying down of the {100} plane and the {110} plane of the face-centered cubic metals parallel to the substrate at higher temperatures as higher energy configurations which are nevertheless stable (i. e., the sign of the sum is negative).

A small square region (15 cell lengths by 15 cell lengths) of silver-chloride lattice imposed on the rock-salt lattice has been mapped. A silver ion was placed directly over a chloride ion at the origin and actual measurement of Ag^+–Cl^- distances, and Ag^+–Na^+ distances was carried out in the first quadrant only. At a cell having coordinates of seven

cell lengths from the origin along both the ordinate and abscissa the misfit has become about 10 percent. The cell at this distance from the origin is shown in figure 9. Each silver ion in the area considered has four chloride ion neighbors and four sodium ion neighbors in the plane below the silver ion monolayer. Planar distances, which were measured, were converted to actual distances by means of a graph of planar distance versus actual distance for which the actual distances were computed by using a 20-inch log-logvector slide rule. The bonding energy is the sum of e^2/r, where the distances, r, are those of the silver ions from their chloride ion neighbors. The antibonding energy is the sum of e^2/r, where the distances, r, are those of the silver ions from their sodium ion neighbors. In the counting process, each distance is counted twice except for the Ag^+–Cl^- distance at the center of the cube face. Where distances were counted twice, half the sum was taken. The result for the first row along the abscissa for seven and a half cells was 7.092 e^2 for the bonding energy, and 4.821 e^2 for the antibonding energy. The net bonding energy was 2.271 e^2. For the seventh row above the abscissa where ~~net~~ the maximum misfit reached about 10 percent, the NET bonding energy had decreased only to 2.259 e^2. At this rate of decrease a very large area could be covered before the antibonding energy became equal to the bonding energy. In this connection, Hass [39] has found that very thin silver films deposited on collodion exist as isolated granules 100 A to 150 A in diameter, but that thin silver films deposited on rock salt do not contract as strongly into isolated granules. Measurement of distances for only the first cell at the origin with the silver chloride turned at 45 degrees with respect to the rock salt cube gave an energy already slightly antibonding. Hence at a nucleus consisting of the first cell, this orientation became impossible.

To find if this basis for predicting epitaxial temperatures has any real value, Suddeth and McCraw have carried out evaporations of metals in a vacuum on rock salt at temperatures which their common ionization potential would indicate to be about correct on the graph of epitaxial temperature against ionization potential (see fig. 8). In the case of lead, which according to its ionization potential should have oriented at about 200° C, a slight orientation was found at 225° C. However, the results with lead were not reproducible even among separate deposits on the same cleavage surface which were made simultaneously and in no case was complete single crystal orientation obtained. It is possible that the misfit was too large and that better results would have been obtained on potassium chloride, but there was no potassium chloride single crystal on hand at the time to use for the purpose. Lead is cubic face-centered with $a=4.93$. Lead chloride is orthorhombic. The cell dimensions are: $a=9.030$, $b=7.608$, and $c=4.525$.

According to the graph of epitaxial temperature against ionization potential, antimony, with a common valence of three, should orient at about 380° C.

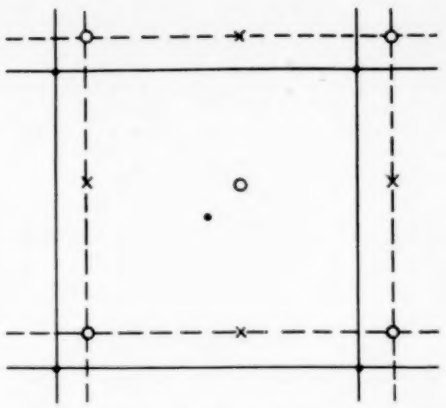

FIGURE 9. *Misfitted silver ion lattice on rock salt.*

●, Silver ion; X, sodium ion; ○, chloride ion

Suddeth and McCraw have found partial orientation for antimony from 250° to 410° C. They found appearances in the diffraction patterns similar to those observed by Brück for aluminum and gold, which he interpreted as the transition from deposition with the {111} face parallel to the substrate to deposition with the {100} face parallel to the substrate. Suddeth and McCraw have not as yet obtained a perfect single crystal of antimony having only the {100} face parallel to the rock salt. Antimony has the arsenic-type lattice. For the antimony cell $a=4.50$, and the nearest neighbor distance d is 2.87. Data for antimony trichloride do not seem to be available. For SbI_3 in a bimolecular cell, $a_0=8.186$ with $\alpha=54°14'$ and in a hexagonal pseudo cell $a'_0=7.466$ and $c'_0=20.89$.

According to the graph of epitaxial temperature against ionization potential, platinum for an ionization state of two should orient at about 250° C. Thirsk [40], by employing evaporation from a hot platinum wire, has obtained orientation of Pt on potassium chloride at the epitaxial temperature of 250° C with the Pt-cube edge parallel to the cube edge of the substrate. The {100} face of potassium chloride was used. The epitaxial temperature on rock salt will not be very different judging from the data of Brück where he used potassium chloride as substrate as well as sodium chloride. The data for the three metals for which he found the epitaxial temperature on both substrates are given in table 4.

TABLE 4

Substrate	Metal deposit	Epitaxial temperature
		°C
NaCl	Ag	150
KCl	Ag	150
NaCl	Pd	250
KCl	Pd	250
NaCl	Ni	370
KCl	Ni	410

Brück pointed out in regard to these data that the temperature varied in the case of nickel where the misfit was large.

Consequently, it seems fairly certain that when an adequate method is used for depositing platinum on rock salt, it will be found to orient at a temperature close to that predicted by the graph of epitaxial temperature against ionization potential.

Suddeth and McCraw made an evaporation of tin at the epitaxial temperature which the graph would predict for the valence state of two. The result was negative with pure tin but showed considerable orientation in an impure sample which probably contained lead.

6. Summary

From an analysis of experimental data on the formation of oriented deposits of metals from their vapors in a vacuum onto salt substrates, it appears reasonable to suppose that a surface ionization process may explain the existence of metal ions that are brought into preferred orientations by attractive electrostatic forces that act between them and the subtrate ions. A relation between ionization potential and the epitaxial temperature at which the {100} plane deposits parallel to the {100} plane of the rock-salt substrate has been found which may be useful as a basis for making predictions as to the temperature at which orientation will occur. The results appear to indicate that the sequence of layers—substrate salt, deposit-metal salt monolayer, deposit-metal—exists.

In making a prediction of a substrate-metal pair and of the temperature at which orientation of the {100} plane of the deposit parallel to the {100} plane of the substrate should occur: (a) It is necessary to consider the lattice fit between the substrate salt and the deposit-metal salt, and the lattice fit between the metal salt and the metal. (Structure data for many of the metal salts in the anhydrous state are at present unavailable.) (b) The epitaxial temperature can be roughly estimated from the accompanying graph by using the most common ionization potential of the metal in question. But if the metal salt corresponding to this ionization potential is unstable, or if this salt cannot orient on the substrate salt, a higher ionization state (and correspondingly higher epitaxial temperature) may be required.

I express my appreciation to L. Marton for his continued interest in this work, and to Robert G. Breckenridge for his suggestions and encouragement.

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Torsion of Anisotropic Elastic Cylinders by Forces Applied on the Lateral Surface[1]

Harold Luxenberg[2]

The classical Saint-Venant theory of torsion presents a method for the determination of the elastic behavior of an orthotropic elastic beam of uniform arbitrary cross section twisted by forces applied on the end sections, with the lateral surface free of stress. In the present paper the theory is extended to include a more generally anisotropic beam (possessing only a single plane of elastic symmetry, rather than three mutually orthogonal planes of elastic symmetry, as in the case of orthotropic materials) built in at one end and twisted by forces uniformly distributed along the lateral surface.

The stresses in a beam of elliptic cross section twisted by constant tangential traction are obtained as an application of the theory.

1. Introduction

The classical Saint-Venant theory of torsion presents a method for the determination of the elastic behavior of an orthotropic elastic beam of uniform arbitrary cross section twisted by forces applied on the end sections, with the lateral surface free of stress. In the present paper the theory is extended to include a more generally anisotropic cylinder (possessing only a single plane of elastic symmetry, rather than three mutually orthogonal planes of elastic symmetry, as in orthotropic materials) built in at one end and twisted by forces uniformly distributed along the lateral surface.

The extension of the Saint-Venant theory of torsion of cylinders by forces applied on the end sections to anisotropic cylinders was made by Lechnitzky [3, 4][3] in 1939.

The torsion of isotropic cylinders by forces applied on the lateral surface has been considered by Filon [1] in 1902, Timpe [11] in 1912, and Zwolinsky and Riz [12] in 1939.

Filon and Timpe showed that the stresses and displacements in an isotropic circular cylinder twisted by forces applied on the lateral surface are derivable from a biharmonic stress function. Zwolinsky and Riz treated the problem of the isotropic cylinder of arbitrary cross section twisted by constant tangential forces, and showed that, in this case, the stresses and displacements are derivable from two stress functions, one biharmonic, the other harmonic. For a circular cylinder the harmonic stress function vanishes, and the Filon-Timpe solution is obtained as a special case of the Zwolinsky-Riz solution.

The results obtained here are more general than those given by Zwolinsky and Riz in two important respects. First, the applied forces assumed in the present treatment are not necessarily constant or tangential to the lateral surface; and second, the results are applicable to a wider range of elastic materials, including wood, which is orthotropic, and certain crystalline media [5, p. 155 to 160], which possess a single plane of elastic symmetry.

It will be noted that no mention has been made of the distribution of stresses over the built-in end of the cylinder. This distribution will, in general, depend on the mode of clamping. If these stresses are prescribed the exact solution of the problem presents considerable difficulty because of the complicated boundary conditions.

In 1855 Saint-Venant stated as a principle of elasticity that two statically equivalent systems of forces applied to a portion of the surface of a medium will produce essentially the same stresses at point of the body not too near the region of application of the forces.

If this principle is accepted, and if it is borne in mind that in most applications only the resultant of the stresses on the end of the cylinder is known. it is seen that a solution of practical usefulness may be obtained by "relaxing" the boundary conditions, and specifying only the resultant stress on the end section.

The extra freedom allowed by the relaxation of the boundary conditions permits us to obtain a solution by the so-called semi-inverse method of Saint-Venant, which consists in making certain assumptions about the stress distributions, while still leaving enough arbitrariness in the assumed stresses to satisfy all the conditions of the problem.

By the use of this method it is shown that the stresses in the twisted cylinder are derivable from two stress functions, $\phi(x,y)$ and $\psi(x,y)$, which satisfy the differential equations.

$$\left.\begin{aligned} l\frac{\partial^2\phi}{\partial x^2}+2m\frac{\partial^2\phi}{\partial x\partial y}+n\frac{\partial^2\phi}{\partial y^2}=0,\\ b_{22}\frac{\partial^4\psi}{\partial x^4}-3b_{26}\frac{\partial^4\psi}{\partial x^3\partial y}+2(b_{12}+b_{66})\frac{\partial^4\psi}{\partial x^2\partial y^2}-\\ 3b_{16}\frac{\partial^4\psi}{\partial x\partial y^3}+b_{11}\frac{\partial^4\psi}{\partial y^4}=b_2\frac{\partial^2\phi}{\partial x^2}-2b_6\frac{\partial^2\phi}{\partial x\partial y}+\\ b_1\frac{\partial^2\phi}{\partial y^2}+b_0 \end{aligned}\right\}(1)$$

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[2] Present address, Remington & Rand, Inc., Philadelphia, Pa.

[3] Figures in brackets indicate the literature references at the end of this paper.

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in the region R corresponding to the interior of the cylinder, and the conditions

$$\left.\begin{aligned}
&\left(l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}-ly+mx\right)\cos(x,\nu)+\\
&\qquad\left(m\frac{\partial\phi}{\partial x}+n\frac{\partial\phi}{\partial y}-my+nx\right)\cos(y,\nu)=0\\
&\frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right)=\left(l\phi+\frac{m}{2}x^2-\frac{N}{\alpha}\right)\cos(x,\nu)+\\
&\qquad\left(\frac{n}{2}x^2-\frac{l}{2}y^2+m\phi+\frac{T}{\alpha}\right)\cos(y,\nu)\\
&-\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right)=\left(\frac{n}{2}x^2-\frac{l}{2}y^2+m\phi-\frac{T}{\alpha}\right)\cos(x,\nu)+\\
&\qquad\left(n\phi-\frac{m}{2}y^2-\frac{N}{\alpha}\right)\cos(y,\nu)
\end{aligned}\right\}\quad(2)$$

on the bounding curve C of each cross section. The parameters l, m, n, b_i, and b_{ij} appearing in these equations are functions of the 13 elastic constants of the material, N and T are the normal and tangential components of the applied surface tractions per unit area, ν designates the unit normal vector to C, and α is given by

$$\alpha=\frac{\int_C(xY-yX)ds}{\int_R\left[nx^2-2mxy+ly^2+x\left(m\frac{\partial\phi}{\partial x}+n\frac{\partial\phi}{\partial y}\right)-y\left(l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}\right)\right]dS}.\quad(3)$$

In eq (3), X and Y are the x and y components of the applied surface traction, and the contour integral is the applied twisting moment per unit length. The surface integral is proportional to the torsional rigidity of the cylinder.

In the case of an isotropic beam the differential equations (1), simplify to

$$\nabla^2\phi=0,\qquad\nabla^4\psi=0,\quad(4)$$

the boundary conditions (2) become

$$\left.\begin{aligned}
&\left(\frac{\partial\phi}{\partial x}-y\right)\cos(x,\nu)+\left(\frac{\partial\phi}{\partial y}+x\right)\cos(y,\nu)=0\\
&\frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right)=\left(\frac{\phi}{2}-\frac{N}{\alpha}\right)\cos(x,\nu)+\\
&\qquad\left(\frac{x^2-y^2}{4}+\frac{T}{\alpha}\right)\cos(y,\nu)\\
&-\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right)=\left(\frac{x^2-y^2}{4}-\frac{T}{\alpha}\right)\cos(x,\nu)+\\
&\qquad\left(\frac{\phi}{2}-\frac{N}{\alpha}\right)\cos(y,\nu)
\end{aligned}\right\}\quad(5)$$

and α is given by

$$\alpha=2\frac{\int_C(xY-yX)ds}{\int_R\left(x^2+y^2+x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)dS}.\quad(6)$$

It is shown, from energy considerations, that the boundary-value problems posed above possess a unique solution, and several methods of solution are discussed.

As an illustration of the method, the stresses in a cylinder of elliptic cross section twisted by constant tangential traction are obtained.

2. General theory

The first fundamental boundary-value problem of elasticity may be stated as follows [8, p. 72]:

Determine the distribution of stress and the displacements in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of the forces acting on the surface of the body is known.

The second fundamental problem is similar, except that in this case the surface displacements rather than the forces acting on the surface are prescribed, and in the third, or mixed, boundary-value problem the forces are prescribed over a portion of the surface and the displacements over the remainder.

The problem of the cylinder that we are considering is of the first kind, where body forces are absent.

In the following paragraphs we shall review certain basic elements of the mathematical theory of elasticity and present the mathematical formulation of the first fundamental boundary-value problem.

The state of stress in an elastic medium is characterized by the six components τ_{xx}, τ_{yy}, τ_{zz}, τ_{yz}, τ_{zx}, τ_{xy}, of the stress tensor. The components of stress, X_ν, Y_ν, Z_ν, on an element of surface whose normal has the direction $\vec{\nu}$ are given in terms of the stress tensor by the relations [8, p. 40]

$$\left.\begin{aligned}
X_\nu&=\tau_{xx}\cos(x,\nu)+\tau_{xy}\cos(y,\nu)+\tau_{zx}\cos(z,\nu)\\
Y_\nu&=\tau_{xy}\cos(x,\nu)+\tau_{yy}\cos(y,\nu)+\tau_{yz}\cos(z,\nu)\\
Z_\nu&=\tau_{zx}\cos(x,\nu)+\tau_{yz}\cos(y,\nu)+\tau_{zz}\cos(z,\nu).
\end{aligned}\right\}\quad(7)$$

The relations (7) may be regarded as defining the six components, τ_{xx}, τ_{yy}, . . ., τ_{xy}, of the stress tensor. If we let the vector $\vec{\nu}$ take on the directions of the three coordinate axes in eq (7), we obtain the usual definition of the stress tensor,

$$\left.\begin{aligned}
X_x&\equiv\tau_{xx},&Y_x&\equiv\tau_{xy},&Z_x&\equiv\tau_{zx}\\
X_y&\equiv\tau_{xy},&Y_y&\equiv\tau_{yy},&Z_y&\equiv\tau_{yz}\\
X_z&\equiv\tau_{zx},&Y_z&\equiv\tau_{yz},&Z_z&\equiv\tau_{zz}.
\end{aligned}\right\}\quad(8)$$

From considerations of static equilibrium it can be shown that

$$X_y = Y_x, \qquad Z_x = X_z, \qquad Y_z = Z_y, \tag{9}$$

and that the defining relations (8) are self-consistent.

For any material in static equilibrium under a prescribed system of surface loading X_ν, Y_ν, Z_ν, and free of body forces, the stress tensor must satisfy the equilibrium equations

$$\left.\begin{aligned} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0 \end{aligned}\right\} \tag{10}$$

in the interior of the medium, and the conditions (7) on the surface, where we now regard X_ν, Y_ν, Z_ν as the prescribed surface tractions.

The strain tensor, e_{xx}, e_{yy}, e_{zz}, e_{yz}, e_{zx}, e_{xy}, is defined in terms of u, v, and w, the x, y, and z components of the displacement vector, by the relations

$$\left.\begin{aligned} e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{zz} = \frac{\partial w}{\partial z}, \\ e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \quad e_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \\ e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \end{aligned}\right\} \tag{11}$$

It may be shown [5, p. 48] that a necessary and sufficient condition that a given tensor may be considered as a strain tensor arising from actual physical displacements in an elastic material is that the tensor satisfy the conditions

$$\left.\begin{aligned} \frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right) \\ \frac{\partial^2 e_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x}\right) \\ \frac{\partial^2 e_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial e_{xy}}{\partial z} + \frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y}\right) \\ 2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} \\ 2\frac{\partial^2 e_{yz}}{\partial y \partial z} = \frac{\partial^2 e_{yy}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial y^2} \\ 2\frac{\partial^2 e_{zx}}{\partial z \partial x} = \frac{\partial^2 e_{zz}}{\partial x^2} + \frac{\partial^2 e_{xx}}{\partial z^2}. \end{aligned}\right\} \tag{12}$$

Equations (12) are known as the compatibility equations and are the integrability conditions for the system of (11), when the displacements u, v, w are to be determined from a given strain tensor.

The stress and strain tensors in an elastic medium are related by the generalized Hooke's law. If the elastic constants of the medium are denoted by c_{ij} $(i,j=1,2, \ldots ,6)$ Hooke's law for a medium in which the xy-plane is a plane of elastic symmetry [4] may be written in the form [8, p. 62]:

$$\left.\begin{aligned} e_{xx} = c_{11}\tau_{xx} + c_{12}\tau_{yy} + c_{13}\tau_{zz} + c_{16}\tau_{xy} \\ e_{yy} = c_{21}\tau_{xx} + c_{22}\tau_{yy} + c_{23}\tau_{zz} + c_{26}\tau_{xy} \\ e_{zz} = c_{31}\tau_{xx} + c_{32}\tau_{yy} + c_{33}\tau_{22} + c_{36}\tau_{xy} \\ e_{yz} = c_{44}\tau_{yz} + c_{45}\tau_{zx} \\ e_{zx} = c_{54}\tau_{yz} + c_{55}\tau_{zx} \\ e_{xy} = c_{61}\tau_{xx} + c_{62}\tau_{yy} + c_{63}\tau_{zz} + c_{66}\tau_{xy} \end{aligned}\right\}, \tag{13}$$

where $c_{12}=c_{21}$, $c_{13}=c_{31}$, $c_{16}=c_{61}$, $c_{23}=c_{32}$, $c_{26}=c_{62}$, $c_{36}=c_{63}$, $c_{45}=c_{54}$.

For isotropic materials, the number of independent elastic constants reduces to two, and Hooke's law has the simpler form

$$\left.\begin{aligned} e_{xx} = \frac{1}{E}\tau_{xx} - \frac{\sigma}{E}(\tau_{yy} + \tau_{zz}), \quad e_{yy} = \frac{1}{E}\tau_{yy} - \\ \frac{\sigma}{E}(\tau_{zz} + \tau_{xx}), \quad e_{zz} = \frac{1}{E}\tau_{zz} - \frac{\sigma}{E}(\tau_{xx} + \tau_{yy}), \\ e_{yz} = \frac{1+\sigma}{E}\tau_{yz}, \quad e_{zx} = \frac{1+\sigma}{E}\tau_{zx}, \quad e_{xy} = \frac{1+\sigma}{E}\tau_{xy}, \end{aligned}\right\} \tag{13a}$$

where E is Young's modulus and σ is Poisson's ratio for the material.

We may now reformulate the first fundamental boundary-value problem of elasticity in mathematical terms:

Determine a set of six functions, the components of the stress tensor, which satisfies the equations (10) in the interior of the elastic medium and the relations (7) on the surface of the medium, where X_ν, Y_ν, and Z_ν are prescribed functions, and such that the strain tensor calculated from these functions by the use of (13) satisfies the compatibility conditions (12).

3. Formulation of the problem

Let the coordinate system be chosen as shown in figure 1, with the built-in end of the cylinder in the

[4] The most general elastic medium is characterized by 21 independent elastic constants. Here we consider a medium whose elastic properties are identical in any two directions symmetric with respect to the xy-plane. This type of elastic symmetry reduces the number of independent elastic constants to 13.

plane $z=0$, and the free end in the plane $z=h$. The boundary curve of any cross section will be denoted by C; the region enclosed by C and the unit normal to C will be denoted by R and $\vec{\nu}$.

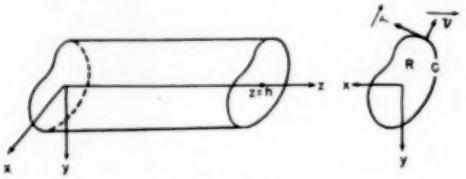

FIGURE 1.

The prescribed surface traction $\vec{T}$ (force per unit area) is assumed to be independent of z, directed parallel to the (x,y) plane, and such that the resultant on any cross section is a twisting moment of magnitude M (torque per unit length).

This condition requires that the components X and Y of the applied surface traction satisfy the relations

$$\int_C X ds=0, \qquad \int_C Y ds=0, \qquad \int_C (xY-yX)ds=M. \tag{14}$$

In order that the first fundamental boundary value problem of elasticity may be formulated in such a way that the solution is unique, it is necessary that the stresses be prescribed over the entire surface, subject, of course, to the conditions of static equilibrium. This would require that Z_x and Z_y, the components of the clamping stresses on the end $z=0$, be prescribed, subject to the conditions

$$\int_R Z_x dS=0, \quad \int_R Z_y dS=0, \quad \int_R (xZ_y-yZ_z)dS=Mh. \tag{15}$$

In this case the boundary conditions on the end sections would be of the form:

$$\left.\begin{aligned} &\text{(a) on the end } z=0\text{:}\\ &\qquad \tau_{zz}=0, \qquad \tau_{xz}=Z_x, \qquad \tau_{yz}=Z_y;\\ &\text{(b) on the end } z=h\text{:}\\ &\qquad \tau_{zz}=0, \qquad \tau_{xz}=0, \qquad \tau_{yz}=0. \end{aligned}\right\} \tag{16}$$

Inasmuch as Z_x and Z_y are unknown (and depend upon the mode of clamping), we shall replace the exact boundary conditions (16) by the relaxed conditions,

$$\left.\begin{aligned} &\text{(a) on the end } z=0\text{:}\\ &\qquad \int_R \tau_{zz}dS=\int_R x\tau_{zz}dS=\int_R y\tau_{zz}dS=0\\ &\qquad \int_R \tau_{xz}dS=\int_R \tau_{yz}dS=0\\ &\qquad \int_R (x\tau_{zy}-y\tau_{zx})dS=Mh;\\ &\text{(b) on the end } z=h\text{:}\\ &\qquad \int_R \tau_{zz}dS=\int_R x\tau_{zz}dS=\int_R y\tau_{zz}dS=0\\ &\qquad \tau_{xz}=0\\ &\qquad \tau_{yz}=0. \end{aligned}\right\} \tag{16a}$$

These are conditions on the resultant stresses, rather than on the stress distributions.

The boundary conditions on the lateral surface, that is, on the curve C, are

$$\left.\begin{aligned} \tau_{xx}\cos(x,\nu)+\tau_{xy}\cos(y,\nu)&=X\\ \tau_{xy}\cos(x,\nu)+\tau_{yy}\cos(y,\nu)&=Y\\ \tau_{zx}\cos(x,\nu)+\tau_{yz}\cos(y,\nu)&=0. \end{aligned}\right\} \tag{17}$$

Our problem, then, is to determine a system of stresses $\tau_{xx}, \tau_{yy}, \ldots, \tau_{xy}$ which satisfies the equilibrium eq (10), the relaxed boundary conditions (16a) and (17), and for which the strain system calculated by the use of Hooke's law (13) satisfies the compatibility equation (12).

The solution of the problem as thus formulated is, of course, not unique, but, according to the principle of Saint-Venant [8, p. 95], the stresses given by any two solutions will be essentially the same at points of the cylinder not too near the end sections.

We shall make use of the Saint-Venant semi-inverse method of solution [8, p. 100]. This consists in making certain assumptions about the stress distribution, and still leaving sufficient freedom to permit the satisfaction of the differential equations and the boundary conditions. If these assumptions are carefully made, a unique solution (unique under these assumptions) exists.

It is apparent that the torsional couple transmitted across each cross section is a linear function of the distance from that cross section to the free end of the cylinder. Hence, it is natural to assume that the stresses in the cylinder are given, in part at least, by a Saint-Venant torsion function.

There is a type of elastic deformation known as plane strain [7, p. 231], in which the displacements are of the form

$$u=u(x,y), \qquad v=v(x,y), \qquad w\equiv 0.$$

The stresses in this type of deformation are derivable from an Airy stress function and can be chosen to satisfy very general boundary conditions of the type given by (17).

We shall assume that the stress system in the twisted cylinder may be derived, in part, from a superposition of the torsion solution and the plane strain type solution.

For a deformation to be of plane strain type, it is necessary that $\tau_{zz} \neq 0$. Because a variable tension on the end sections may produce a bending or extension of the cylinder, we shall further superpose the bending and tension type solutions.

Thus, if we denote by τ_{ij} any one of the six components of the stress tensor, we may write

$$\frac{\tau_{ij}}{\alpha} = \tau_{ij}^{(1)} + \tau_{ij}^{(2)} + \tau_{ij}^{(3)}, \tag{18}$$

where α is a constant to be determined by conditions of static equilibrium, $\tau_{ij}^{(1)}$ is derived from an Airy stress function, $\tau_{ij}^{(2)}$ is derived from the Saint-Venant torsion function, and $\tau_{ij}^{(3)}$ is required to satisfy the condition that the end surfaces be free of stresses that would produce bending moments and extensions.

The stress system $\tau_{ij}^{(1)}$ may be written [7, p. 231]

$$\left.\begin{aligned} &\tau_{xx}^{(1)} = -\frac{\partial^2\psi}{\partial x^2}, \quad \tau_{yy}^{(1)} = -\frac{\partial^2\psi}{\partial y^2} \\ &\tau_{xy}^{(1)} = \frac{\partial^2\psi}{\partial x \partial y}, \quad \tau_{zz}^{(1)} = 0, \quad \tau_{yz}^{(1)} = 0 \\ &\tau_{xz}^{(1)} = p\frac{\partial^2\psi}{\partial x^2} + q\frac{\partial^2\psi}{\partial x \partial y} + r\frac{\partial^2\psi}{\partial y^2}, \end{aligned}\right\} \tag{19}$$

where $\psi[\equiv\psi(x,y)]$ is the Airy stress function and p, q, and r, are functions of the elastic constants. We note that the stress system $\tau_{ij}^{(1)}$ satisfies the equilibrium equation (10) identically.

The stress system $\tau_{ij}^{(3)}$ has the form [8, p. 101, 108]

$$\tau_{xz}^{(3)} = A + Bx + Cy, \qquad \tau_{xx}^{(3)} = \tau_{yy}^{(3)} = \tau_{xy}^{(3)} = \tau_{zz}^{(3)} = \tau_{yz}^{(3)} = 0, \tag{20}$$

where A, B, C are determined by the three relations

$$\int_R \tau_{zz}\,dS = 0, \qquad -\int_R x\tau_{zz}\,dS = 0, \qquad \int_R y\tau_{zz}\,dS = 0, \tag{21}$$

which state that the resultant bending couples and tension vanish on the end sections.

The system $\tau_{ij}^{(2)}$ is a modification of the Saint-Venant stress system τ'_{ij}. Sokolnikoff [8, p. 214] gives the stresses τ'_{ij} in an orthotropic beam twisted by end couples as

$$\left.\begin{aligned} &\tau'_{yz} = n\left(\frac{\partial\phi}{\partial y} + x\right) + m\left(\frac{\partial\phi}{\partial x} - y\right) \\ &\tau'_{xz} = l\left(\frac{\partial\phi}{\partial x} - y\right) + m\left(\frac{\partial\phi}{\partial y} + x\right) \\ &\tau'_{xx} = \tau'_{yy} = \tau'_{zz} = \tau'_{xy} = 0, \end{aligned}\right\} \tag{22}$$

where l, m, n are functions of the elastic constants and $\phi[\equiv\phi(x,y)]$ is the torsion function. Because we wish the torsional couple to vanish on the face $z=h$, we shall set

$$\left.\begin{aligned} \tau_{yz}^{(2)} &= (h-z)\left\{n\left(\frac{\partial\phi}{\partial y} + x\right) + m\left(\frac{\partial\phi}{\partial x} - y\right)\right\} \\ \tau_{xz}^{(2)} &= (h-z)\left\{l\left(\frac{\partial\phi}{\partial x} - y\right) + m\left(\frac{\partial\phi}{\partial y} + x\right)\right\}. \end{aligned}\right\} \tag{23}$$

The third of the equilibrium equations (10) will be satisfied if $\phi(x,y)$ satisfies the condition

$$l\frac{\partial^2\phi}{\partial x^2} + 2m\frac{\partial^2\phi}{\partial x \partial y} + n\frac{\partial^2\phi}{\partial y^2} = 0. \tag{24}$$

In order that the first two equilibrium equations in the set (10) be satisfied identically, we set

$$\tau_{xx}^{(2)} = l\phi + \frac{m}{2}x^2, \qquad \tau_{yy}^{(2)} = n\phi - \frac{m}{2}y^2,$$

$$\tau_{xy}^{(2)} = \frac{n}{2}x^2 - \frac{l}{2}y^2 + m\phi. \tag{25}$$

To simplify the satisfaction of the compatibility conditions (12), we choose

$$\tau_{zz}^{(3)} = -\phi + \frac{s}{2}x^2 - \frac{t}{2}y^2, \tag{26}$$

with s and t as yet undetermined.

Our assumed stress system, then, is the following:

$$\left.\begin{aligned} &\frac{1}{\alpha}\tau_{xx} = l\phi - \frac{\partial^2\psi}{\partial y^2} + \frac{m}{2}x^2 \\ &\frac{1}{\alpha}\tau_{yy} = n\phi - \frac{\partial^2\psi}{\partial x^2} - \frac{m}{2}y^2 \\ &\frac{1}{\alpha}\tau_{zz} = p\frac{\partial^2\psi}{\partial x^2} + q\frac{\partial^2\psi}{\partial x \partial y} + r\frac{\partial^2\psi}{\partial y^2} - \phi + \frac{s}{2}x^2 - \frac{t}{2}y^2 + A + Bx + Cy \\ &\frac{1}{\alpha}\tau_{yz} = (h-z)\left\{n\left(\frac{\partial\phi}{\partial y} + x\right) + m\left(\frac{\partial\phi}{\partial x} - y\right)\right\} \\ &\frac{1}{\alpha}\tau_{xz} = (h-z)\left\{l\left(\frac{\partial\phi}{\partial x} - y\right) + m\left(\frac{\partial\phi}{\partial y} + x\right)\right\} \\ &\frac{1}{\alpha}\tau_{xy} = \frac{n}{2}x^2 - \frac{l}{2}y^2 + \frac{\partial^2\psi}{\partial x \partial y} + m\phi, \end{aligned}\right\} \tag{27}$$

with the constants α, l, m, n, p, q, r, s, and t as yet undetermined.

We next determine p, q, r, s, and t, in such a way that e_{zz} will be independent of $\psi(x,y)$ and of the terms in x^2 and y^2. The insertion of the stresses given by (27) into the third of (13) leads to the following expression for e_{zz}:

$$\left.\begin{aligned}\frac{1}{\alpha}e_{zz}=&(c_{31}l+c_{32}n-c_{33}+c_{36}m)\phi+\\&c_{33}(A+Bx+Cy)+\\&\frac{1}{2}(c_{31}m+c_{33}s+c_{36}n)x^2-\\&\frac{1}{2}(c_{32}m+c_{33}t+c_{36}l)y^2+(c_{33}p-c_{32})\frac{\partial^2\psi}{\partial x^2}+\\&(c_{22}q+c_{36})\frac{\partial^2\psi}{\partial x\partial y}+(c_{33}r-c_{31})\frac{\partial^2\psi}{\partial y^2}.\end{aligned}\right\}\quad(28)$$

If we set

$$p=c_{32}/c_{33},\quad q=-c_{36}/c_{33},\quad r=c_{31}/c_{33},$$
$$s=-(c_{31}m+c_{36}n)/c_{33},\quad t=-(c_{32}m+c_{36}l)/c_{33},\qquad(29)$$

the last five terms in the right-hand member of (28) will vanish.

If the values for p, q, r, s, and t given in (29) are substituted in the stress equations (27), and the strains are then calculated from Hooke's law (13), we have the following expressions for the components of the strain tensor:

$$\left.\begin{aligned}\frac{1}{\alpha}e_{xx}=&b_1\phi-b_{12}\frac{\partial^2\psi}{\partial x^2}+b_{16}\frac{\partial^2\psi}{\partial x\partial y}-b_{11}\frac{\partial^2\psi}{\partial y^2}+\\&(b_{11}m+b_{16}n)\frac{x^2}{2}-(b_{16}l+b_{12}m)\frac{y^2}{2}+\\&c_{13}(A+Bx+Cy)\\\frac{1}{\alpha}e_{yy}=&b_2\phi-b_{22}\frac{\partial^2\psi}{\partial x^2}+b_{26}\frac{\partial^2\psi}{\partial x\partial y}-b_{21}\frac{\partial^2\psi}{\partial y^2}+\\&(b_{21}m+b_{26}n)\frac{x^2}{2}-(b_{26}l+b_{22}m)\frac{y^2}{2}+\\&c_{23}(A+Bx+Cy)\\\frac{1}{\alpha}e_{zz}=&b_3\phi+c_{33}(A+Bx+Cy)\\\frac{1}{\alpha}e_{yz}=&(h-z)\left\{(c_{44}n+c_{45}m)\left(\frac{\partial\phi}{\partial y}+x\right)+\right.\\&\left.(c_{44}m+c_{45}l)\left(\frac{\partial\phi}{\partial x}-y\right)\right\}\\\frac{1}{\alpha}e_{xz}=&(h-z)\left\{(c_{54}m+c_{55}l)\left(\frac{\partial\phi}{\partial x}-y\right)+\right.\\&\left.(c_{54}n+c_{55}m)\left(\frac{\partial\phi}{\partial y}+x\right)\right\}\end{aligned}\right\}\quad(30)$$

$$\left.\begin{aligned}\frac{1}{\alpha}e_{xy}=&b_6\phi-b_{62}\frac{\partial^2\psi}{\partial x^2}+b_{66}\frac{\partial^2\psi}{\partial x\partial y}-b_{61}\frac{\partial^2\psi}{\partial y^2}+\\&(b_{61}m+b_{66}n)\frac{x^2}{2}-(b_{66}l+b_{62}m)\frac{y^2}{2}+\\&c_{63}(A+Bx+Cy),\end{aligned}\right\}\quad(30)\text{ Con.}$$

where

$$\left.\begin{aligned}b_{ij}=b_{ji}=&\begin{vmatrix}c_{ij}c_{i3}\\c_{3j}c_{33}\end{vmatrix}\Big/c_{33},\quad i,j=1,2,6\\b_i=&c_{i1}l+c_{i2}n-c_{i3}+c_{i6}m,\quad i=1,2,3,6.\end{aligned}\right\}\quad(31)$$

If the components of the train tensor given by (30) are inserted in the compatibility equations (12), it is seen that all but the fourth of the compatibility equations are satisfied by the assumed stresses if l, m, and n can be chosen to satisfy the six equations

$$\left.\begin{aligned}c_{44}n-c_{55}l&=0\\c_{45}l+c_{44}m&=0\\c_{45}n+c_{55}m&=0\\l(c_{55}+c_{31})+m(2c_{45}+c_{36})+n(c_{44}+c_{32})&=c_{33}\\l(c_{31})+m(2c_{45}+c_{36})+n(2c_{44}+c_{32})&=c_{33}\\l(2c_{55}+c_{31})+m(2c_{45}+c_{36})+n(c_{32})&=c_{33}.\end{aligned}\right\}\quad(32)$$

It is easily seen that only three of these six equations are linearly independent and that the compatibility equations are satisfied if we set

$$l=c_{33}c_{44}/\beta,\qquad m=-c_{33}c_{45}/\beta,\qquad n=c_{33}c_{55}/\beta,\quad(33)$$

where

$$\beta=c_{44}(c_{55}+c_{31})+c_{55}(c_{44}+c_{32})-c_{45}(2c_{45}+c_{36}).$$

For convenience in calculation we rewrite the expressions for s and t given by (29) in terms of the elastic constants, using the relations (33):

$$s=(c_{31}c_{45}-c_{36}c_{55})/\beta,\qquad t=(c_{32}c_{45}-c_{36}c_{44})/\beta.\quad(34)$$

The fourth of the compatibility equations (12), the only one that is dependent on $\psi(x,y)$, requires that $\psi(x,y)$ satisfy the relation

$$\begin{aligned}b_{22}\frac{\partial^4\psi}{\partial x^4}-3b_{26}\frac{\partial^4\psi}{\partial x^3\partial y}+2(b_{12}+b_{66})\frac{\partial^4\psi}{\partial x^2\partial y^2}-\\3b_{16}\frac{\partial^4\psi}{\partial x\partial y^3}+b_{11}\frac{\partial^4\psi}{\partial y^4}=b_2\frac{\partial^2\phi}{\partial x^2}-2b_6\frac{\partial^2\phi}{\partial x\partial y}+\\b_1\frac{\partial^2\phi}{\partial y^2}+b_{26}n-b_{16}l.\end{aligned}\quad(35)$$

In the next section we shall consider the boundary conditions on the functions $\phi(x,y)$ and $\psi(x,y)$.

4. Boundary conditions

It has been shown in the previous section that the system of stresses given by

$$\left.\begin{aligned}
&\frac{1}{\alpha}\tau_{zz}=l\phi-\frac{\partial^2\psi}{\partial y^2}+\frac{m}{2}x^2\\
&\frac{1}{\alpha}\tau_{yy}=n\phi-\frac{\partial^2\psi}{\partial x^2}-\frac{m}{2}y^2\\
&\frac{1}{\alpha}\tau_{xx}=\frac{c_{32}}{c_{33}}\frac{\partial^2\psi}{\partial x^2}-\frac{c_{36}}{c_{33}}\frac{\partial^2\psi}{\partial x\partial y}+\frac{c_{31}}{c_{33}}\frac{\partial^2\psi}{\partial y^2}+\\
&\qquad\frac{c_{31}c_{45}-c_{36}c_{55}}{2\beta}x^2-\frac{c_{32}c_{45}-c_{36}c_{44}}{2\beta}y^2-\phi+\\
&\qquad A+Bx+Cy\\
&\frac{1}{\alpha}\tau_{yz}=(h-z)\left\{n\left(\frac{\partial\phi}{\partial y}+x\right)+m\left(\frac{\partial\phi}{\partial x}-y\right)\right\}\\
&\frac{1}{\alpha}\tau_{zx}=(h-z)\left\{l\left(\frac{\partial\phi}{\partial x}-y\right)+m\left(\frac{\partial\phi}{\partial y}+x\right)\right\}\\
&\frac{1}{\alpha}\tau_{xy}=\frac{n}{2}x^2-\frac{l}{2}y^2+\frac{\partial^2\psi}{\partial x\partial y}+m\phi,
\end{aligned}\right\}\quad(36)$$

will satisfy the equilibrium equations (10) if $\phi(x,y)$ satisfies

$$l\frac{\partial^2\phi}{\partial x^2}+2m\frac{\partial^2\phi}{\partial x\partial y}+n\frac{\partial^2\phi}{\partial y^2}=0,\qquad(37)$$

and will satisfy the compatibility conditions (12), provided that

$$\left.\begin{aligned}
&l=c_{33}c_{44}/\beta,\qquad m=-c_{33}c_{45}/\beta,\qquad n=c_{33}c_{55}/\beta\\
&\beta=c_{44}(c_{55}+c_{31})+c_{55}(c_{44}+c_{32})-c_{45}(2c_{45}+c_{36}),
\end{aligned}\right\}\quad(38)$$

and $\psi(x,y)$ satisfies the differential equation

$$\begin{aligned}
b_{22}\frac{\partial^4\psi}{\partial x^4}-3b_{28}\frac{\partial^4\psi}{\partial x^3\partial y}+2(b_{12}+b_{66})\frac{\partial^4\psi}{\partial x^2\partial y^2}-\\
3b_{16}\frac{\partial^4\psi}{\partial x\partial y^3}+b_{11}\frac{\partial^4\psi}{\partial y^4}=b_2\frac{\partial_2\phi}{\partial x^2}-2b_6\frac{\partial^2\phi}{\partial x\partial y}+\\
b_1\frac{\partial^2\phi}{\partial y^2}+b_0,
\end{aligned}\qquad(39)$$

where

$$b_{ij}=b_{ji}=\left|\begin{matrix}c_{if}c_{jf}\\c_{3f}c_{33}\end{matrix}\right|/c_{33}\qquad i,j=1,2,6,$$

and

$$\left.\begin{aligned}
&b_i=c_{i1}l+c_{i2}n-c_{i3}+c_{i6}m,\qquad i=1,2,3,6.\\
&b_0=b_{26}n-b_{16}l.
\end{aligned}\right\}\quad(40)$$

Since both of the functions $\phi(x,y)$ and $\psi(x,y)$ appear in (39), it is apparent that we must first determine $\phi(x,y)$. Fortunately the third boundary condition of the set (17) is independent of $\psi(x,y)$, so that the two boundary value problems are separable.

If the values of τ_{yz} and τ_{zx} given by (36) are inserted into the third of (17), we see that this boundary condition will be satisfied if

$$\begin{aligned}
\left\{l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}\right\}\cos(x,\nu)+\left\{m\frac{\partial\phi}{\partial x}+n\frac{\partial\phi}{\partial y}\right\}\cos(y,\nu)=\\
(ly-mx)\cos(x,\nu)+(my-nx)\cos(y,\nu)
\end{aligned}\qquad(41)$$

on C. It is shown by Sokolnikoff [8, p. 215] that the change of variables

$$\xi=\sqrt{ln-m^2}\,x,\qquad\eta=y-\frac{m}{l}x,$$

$$\phi'(\xi,\eta)=\frac{\sqrt{ln-m^2}}{l}\phi(\xi,n)\qquad(42)$$

will reduce the systems (37) and (41) to the simpler system

$$\frac{\partial^2\phi'(\xi,\eta)}{\partial\xi^2}+\frac{\partial^2\phi'(\xi,\eta)}{\partial n^2}=0\qquad(43)$$

$$\frac{d}{d\nu}\phi'(\xi,\eta)=\eta\cos(\xi,\nu)-\xi\cos(\eta,\nu)=0\text{ on }C'.$$

The problem of determining $\phi(x,y)$ is thus reducible to a problem of Neumann. Since the right hand member of the second of equations (43) is a perfect differential, the determination of $\phi'(\xi,\eta)$ is possible, and is unique to within an additive constant.

It should be pointed out that the quantity $(ln-m^2)$ is positive for all elastic media, since from the defining relations (33), we observe that

$$ln-m^2\equiv\frac{c_{33}^2}{\beta^2}(c_{44}c_{55}-c_{45}^2)\qquad(44)$$

and the positiveness of $(c_{44}c_{45}-c_{45}^2)$ is guaranteed by the positive definite nature of the strain energy form.[5] Once the function $\phi(x,y)$ has been determined the right-hand member of (39) is a known function.

Let us express the surface tractions X and Y in terms of the tangential traction T and the normal traction N, by the relations

$$\begin{aligned}
X=N\cos(x,\nu)-T\cos(y,\nu),\\
Y=N\cos(y,\nu)+T\cos(x,\nu).
\end{aligned}\qquad(45)$$

[5] See appendix.

If we make use of the relations (45), the stresses given by (36), and the relations

$$\cos(x,\nu)=\frac{dy}{ds},\qquad \cos(y,\nu)=-\frac{dx}{ds},\tag{46}$$

the first two equations of (17) may be rewritten in the form

$$\left.\begin{aligned}\frac{d}{ds}\left(\frac{\partial\psi}{dy}\right)&=\left(l\phi+\frac{m}{2}x^2-\frac{N}{\alpha}\right)\cos(x,\nu)+\\&\qquad\left(\frac{n}{2}x^2-\frac{l}{2}y^2+m\phi+\frac{T}{\alpha}\right)\cos(y,\nu),\\-\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right)&=\left(\frac{n}{2}x^2-\frac{l}{2}y^2+m\phi-\frac{T}{\alpha}\right)\cos(x,\nu)+\\&\qquad\left(n\phi-\frac{m}{2}y^2-\frac{N}{\alpha}\right)\cos(y,\nu),\end{aligned}\right\}\tag{47}$$

on C. Let us denote the right-hand members of these equations by $F_2(x,y)$ and $F_1(x,y)$, respectively. If we can find a particular (single-valued) solution, $\psi^*(x,y)$ of (39), we must then solve the homogeneous system:

$$\left.\begin{aligned}b_{22}\frac{\partial^4\psi}{\partial x^4}-3b_{26}\frac{\partial^4\psi}{\partial x^3\partial y}+2(b_{12}+b_{66})\frac{\partial^4\psi}{\partial x^2\partial y^2}-\\3b_{16}\frac{\partial^4\psi}{\partial x\partial y^3}+b_{11}\frac{\partial^4\psi}{\partial y^4}=0\\\frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right)=-\frac{d}{ds}\left(\frac{\partial\psi^*}{\partial y}\right)+F_2(x,y)\\-\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right)=\frac{d}{ds}\left(\frac{\partial\psi^*}{\partial x}\right)+F_1(x,y).\end{aligned}\right\}\tag{48}$$

It is easily deduced from the results given by Sokolnikof [7, sections 66 and 95] in his discussion of the theory of anisotropic plates that the system (48) possesses an essentially unique, single-valued solution $\psi(x,y)$, provided that (a) the roots of the characteristic equation

$$b_{22}\lambda^4-3b_{26}\lambda^3+2(b_{12}+b_{66})\lambda^2-3b_{16}\lambda+b_{11}=0\tag{49}$$

are all complex, and (b)

$$\int_C F_2ds=0,\qquad \int_C F_1ds=0,\qquad \int_C(xF_1-yF_2)ds=0.\tag{50}$$

It was shown by Lechnitzky[6] [3] and [4], through energy considerations that the roots of eq (49) are always complex. Thus condition (a) is fulfilled.

We shall test the functions $F_1(x,y)$ and $F_2(x,y)$, in the first two equations of (50). These equations become

$$\left.\begin{aligned}\int_C\left(l\phi+\frac{m}{2}x^2\right)dy-\left(\frac{n}{2}x^2-\frac{l}{2}y^2+m\phi\right)dx\\=\frac{1}{\alpha}\int_C\{N\cos(x,\nu)-T\cos(y,\nu)\}ds,\\\int_C\left(\frac{n}{2}x^2-\frac{l}{2}y^2+m\phi\right)dy-\left(n\phi-\frac{m}{2}y^2\right)dx\\=\frac{1}{\alpha}\int_C N\cos(y,\nu)+T\cos(x,\nu)ds.\end{aligned}\right\}\tag{51}$$

The right-hand members, by the use of eq (45) are seen to represent the components of the resultant force per unit length.

By making use of Green's theorem,

$$\int_C Ndy-Mdx=\int_R\left(\frac{\partial N}{\partial x}+\frac{\partial M}{\partial y}\right)dS,\tag{52}$$

we rewrite the left-hand members of (51) in the forms

$$\begin{aligned}&\int_R\left\{l\left(\frac{\partial\phi}{\partial x}-y\right)+m\left(\frac{\partial\phi}{\partial y}+x\right)\right\}dS,\\&\int_R\left\{m\left(\frac{\partial\phi}{\partial x}-y\right)+m\left(\frac{\partial\phi}{\partial y}+x\right)\right\}dS.\end{aligned}\tag{53}$$

Since $\phi(x,y)$ satisfies the differential equation (37)

$$l\frac{\partial^2\phi}{\partial x^2}+2m\frac{\partial^2\phi}{\partial x\partial y}+n\frac{\partial^2\phi}{\partial y^2}=0,$$

we may rewrite the expressions (53) in the forms

$$\begin{aligned}&\int_R\left[\frac{\partial}{\partial x}\left\{xl\left(\frac{\partial\phi}{\partial x}-y\right)+xm\left(\frac{\partial\phi}{\partial y}+x\right)\right\}+\right.\\&\qquad\left.\frac{\partial}{\partial y}\left\{xm\left(\frac{\partial\phi}{\partial x}-y\right)+xn\left(\frac{\partial\phi}{\partial y}+x\right)\right\}\right]dS\\&\int_R\left[\frac{\partial}{\partial y}\left\{yn\left(\frac{\partial\phi}{\partial y}+x\right)+ym\left(\frac{\partial\phi}{\partial x}-y\right)\right\}+\right.\\&\qquad\left.\frac{\partial}{\partial x}\left\{ym\left(\frac{\partial\phi}{\partial y}+x\right)+yl\left(\frac{\partial\phi}{\partial x}-y\right)\right\}\right]dS.\end{aligned}\tag{54}$$

Once more we use Green's theorem (52), this time on the expressions (54). These now have the forms

$$\begin{aligned}&\int_C x\left[\left\{l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}-ly+mx\right\}\cos(x,\nu)+\right.\\&\qquad\left.\left\{m\frac{\partial\phi}{\partial x}+n\frac{\partial\phi}{\partial y}-my+nx\right\}\cos(y,\nu)\right]ds\\&\int_C y\left[\left\{l\frac{\partial\phi}{\partial x}+m\frac{\partial\phi}{\partial y}-ly+mx\right\}\cos(x,\nu)+\right.\\&\qquad\left.\left\{m\frac{\partial\phi}{\partial x}+n\frac{\partial\phi}{\partial y}-my+nx\right\}\cos(x,\nu)\right]ds.\end{aligned}\tag{55}$$

[6] Lechnitzky considered a similar equation, arising in the study of anisotropic plates, whose roots are the negatives of the roots of eq (49).

The expressions in brackets vanish, by (41), and hence the first two of our integrability equation (50) become, in this case:

$$\int_C X ds = 0, \qquad \int_C Y ds = 0.$$

These are the conditions which the prescribed components X and Y of the applied stress were assumed to satisfy in section 3, hence the first two integrability conditions (50) are satisfied.

Let us now consider the third equation of (50). We express N and T in terms of X and Y and see that we must have

$$\frac{1}{\alpha}\int_C (xY - yX)ds = \int_C \left\{\frac{n}{2}x^2 y - \frac{l}{2}y^3 + my\phi - xn\phi + \frac{m}{2}xy^2\right\}dx + \left\{\frac{n}{2}x^3 - \frac{l}{2}xy^2 + mx\phi - yl\phi - \frac{m}{2}yx^2\right\}dy. \quad (56)$$

We apply Green's theorem to the right-hand member of (56), to obtain

$$\frac{1}{\alpha}\int_C (xY - yX)ds = \int_R \left[nx^2 - 2mxy + ly^2 + x\left(m\frac{\partial\phi}{\partial x} + n\frac{\partial\phi}{\partial y}\right) - y\left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y}\right)\right]dS. \quad (57)$$

Now, the left-hand member is proportional to the twisting moment per unit length, M, that is,

$$M = \int_C (xY - yX)ds. \quad (58)$$

Let us denote by D' the expression [7]

$$D' = \int_R \left[nx^2 - 2mxy + ly^2 + x\left(m\frac{\partial\phi}{\partial x} + n\frac{\partial\phi}{\partial y}\right) - y\left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y}\right)\right]dS. \quad (59)$$

Then, if α is taken as

$$\alpha = M/D', \quad (60)$$

the third equation of (50) is satisfied, and a single-valued solution of (48) exists.

We have satisfied the boundary conditions on the lateral surface; next we shall consider the relaxed boundary conditions on the end sections.

On both end sections we require that there be no resultant forces. Hence we require that

$$\int_R \tau_{zz} dS = 0, \qquad \int_R \tau_{zz} dS = 0, \qquad \int_R \tau_{zy} dS = 0.$$

The first of these conditions can be satisfied by adjusting the parameters A, B, and C. The second and third are easily shown to hold by considering (53) through (55).

On both ends we require that the bending couples, $\overset{z}{M}_x$ and $\overset{z}{M}_y$, vanish. These requirements can be met by adjusting the parameters A, B and C, so that

$$\overset{z}{M}_x \equiv \int_R y\tau_{zz} dS = 0, \qquad \overset{z}{M}_y \equiv -\int_R x\tau_{zz} dS = 0. \quad (61)$$

On the end $z=0$, we require that $\overset{z}{M}_z = M \cdot h$ and on the end $z=h$, we must have $\overset{z}{M}_z = 0$. The second condition is easily seen to be satisfied since $\tau_{zz} = \tau_{yz} = 0$ for $z=h$.

Since $\overset{z}{M}_z$ for $z=0$ is given by

$$\overset{z}{M}_z = h\alpha \int_R \left\{x\left[n\left(\frac{\partial\phi}{\partial y} + x\right) + m\left(\frac{\partial\phi}{\partial x} - y\right)\right] - y\left[l\left(\frac{\partial\phi}{\partial x} - y\right) + m\left(\frac{\partial\phi}{\partial y} + x\right)\right]\right\}dS, \quad (62)$$

and the integrand is equal to the integrand of (57), it follows that

$$\overset{z}{M}_z = h\alpha D' = hM. \quad (63)$$

Thus the stresses given by (36) can be made to satisfy the boundary conditions (16a) and (17), the conditions of static equilibrium (10) and the compatibility conditions (12) by proper choice of the constants l, m, and n, and the functions $\phi(x,y)$ and $\psi(x,y)$.

In the next section we shall specialize the equations to the isotropic case, and in section 6 we shall illustrate the use of the method by calculating the stresses in an elliptic cylinder twisted by constant tangential traction.

5. Isotropic beam

If the beam is isotropic the differential equations and boundary conditions are considerably simplified. In this case the elastic constants c_{ij} may be expressed in terms of E, the Young's modulus, and σ, the Poisson ratio, as follows.

$$\left.\begin{aligned} &c_{11} = c_{22} = c_{33} = \frac{1}{E} \\ &c_{23} = c_{32} = c_{21} = c_{12} = c_{13} = c_{31} = -\frac{\sigma}{E} \\ &c_{44} = c_{55} = c_{66} = \frac{1+\sigma}{E} \\ &c_{16} = c_{61} = c_{26} = c_{62} = c_{36} = c_{63} = c_{45} = c_{54} = 0. \end{aligned}\right\} \quad (64)$$

[7] The quantity D' is proportional to the Saint-Venant coefficient of torsional rigidity D, which arises in the theory of torsion by end couples.

The elastic constants l, m, n, b_{i}, b_{ij} are given by

$$\left.\begin{aligned} &l=n=\tfrac{1}{2}\\ &m=0\\ &b_{11}=b_{22}=(1-\sigma^2)/E\\ &b_{66}=(1+\sigma)/E\\ &b_{12}=-(\sigma+\sigma^2)/E\\ &b_{16}=b_{26}=0\\ &b_1=b_2=-b_3=(1+\sigma)/E\\ &b_6=0. \end{aligned}\right\} \tag{65}$$

The function $\phi(x,y)$ must satisfy the conditions

$$\left.\begin{aligned} &\nabla^2\phi(x,y)=0\\ &\frac{d\phi}{dn}=y\cos(x,\nu)-x\cos(y,\nu),\ \text{on } C \end{aligned}\right\} \tag{66}$$

and $\psi(x,y)$ must satisfy

$$\left.\begin{aligned} &\nabla^4\psi(x,y)=0\\ &\frac{d}{ds}\left(\frac{\partial\psi}{\partial y}\right)=\left(\frac{\phi}{2}-\frac{N}{\alpha}\right)\cos(x,\nu)+\left(\frac{x^2-y^2}{4}+\frac{T}{\alpha}\right)\cos(y,\nu)\\ &-\frac{d}{ds}\left(\frac{\partial\psi}{\partial x}\right)=\left(\frac{x^2-y^2}{4}-\frac{T}{\alpha}\right)\cos(x,\nu)+\left(\frac{\phi}{2}-\frac{N}{\alpha}\right)\cos(y,\nu). \end{aligned}\right\} \tag{67}$$

The torsional rigidity D' is calculated from the defining equation

$$D'=\frac{1}{2}\int_R\left[x^2+y^2+x\frac{\partial\psi}{\partial y}-y\frac{\partial\phi}{\partial x}\right]dxdy \tag{68}$$

and is related to the Saint-Venant coefficient of torsional rigidity D by the relation

$$D'=D(1+\sigma)/E. \tag{69}$$

The stress system in this case has the simpler form,

$$\left.\begin{aligned} &\frac{1}{\alpha}\tau_{xx}=\frac{\phi}{2}-\frac{\partial^2\psi}{\partial y^2}\\ &\frac{1}{\alpha}\tau_{yy}=\frac{\phi}{2}-\frac{\partial^2\psi}{\partial x^2}\\ &\frac{1}{\alpha}\tau_{zz}=-\sigma\nabla^2\psi-\phi+(A+Bx+Cy)\\ &\frac{1}{\alpha}\tau_{yz}=\frac{1}{2}(h-z)\left(\frac{\partial\phi}{\partial y}+x\right)\\ &\frac{1}{\alpha}\tau_{zx}=\frac{1}{2}(h-z)\left(\frac{\partial\phi}{\partial x}-y\right)\\ &\frac{1}{\alpha}\tau_{xy}=\frac{1}{4}(x^2-y^2)+\frac{\partial^2\psi}{\partial x\partial y}, \end{aligned}\right\} \tag{70}$$

Con.

and the strains are given by

$$\left.\begin{aligned} &\frac{1}{\alpha}e_{xx}=\frac{1+\sigma}{2E}\phi+\frac{\sigma+\sigma^2}{E}\frac{\partial^2\psi}{\partial x^2}-\frac{1-\sigma^2}{E}\frac{\partial^2\psi}{\partial y^2}-\frac{\sigma}{E}(A+Bx+Cy)\\ &\frac{1}{\alpha}e_{yy}=\frac{1+\sigma}{2E}\phi-\frac{1-\sigma^2}{E}\frac{\partial^2\psi}{\partial x^2}+\frac{\sigma+\sigma^2}{E}\frac{\partial^2\psi}{\partial y^2}-\frac{\sigma}{E}(A+Bx+Cy)\\ &\frac{1}{\alpha}e_{zz}=-\frac{1+\sigma}{E}\phi+\frac{1}{E}(A+Bx+Cy)\\ &\frac{1}{\alpha}e_{yz}=\frac{1+\sigma}{2E}(h-z)\left(\frac{\partial\psi}{\partial y}+x\right)\\ &\frac{1}{\alpha}e_{zx}=\frac{1+\sigma}{2E}(h-z)\left(\frac{\partial\psi}{\partial x}-y\right)\\ &\frac{1}{\alpha}e_{xy}=\frac{1+\sigma}{E}\frac{\partial^2\psi}{\partial y\partial x}+\frac{1+\sigma}{4E}(x^2-y^2). \end{aligned}\right\} \tag{71}$$

6. Stresses in an elliptic cylinder

As an illustration of the theory developed in the preceding sections, we shall calculate the stresses in an anisotropic cylinder of elliptic cross section twisted by a constant tangential traction T.

Let the equation of the elliptic boundary, C, of the cross section of the cylinder be given in the parametric form

$$x=a\cos\theta,\qquad y=b\sin\theta. \tag{72}$$

Then,

$$\cos(x,\nu):\cos(y,\nu)=b\cos\theta:a\sin\theta. \tag{73}$$

Let us set

$$\phi(x,y)=\sum_{i+j\leq 2}A_{ij}x^iy^j,\qquad i,j=0,1,2. \tag{74}$$

and

$$\psi(x,y)=\sum_{i+j\leq 4}B_{ij}x^iy^j,\qquad i,j=0,1,2,3,4, \tag{75}$$

and seek to determine the coefficients A_{ij} and B_{ij} in such a way that (37), (39), (41) and (47) are satisfied.

We first calculate M from the defining relation (58),

$$M=\int_C(xY-yX)\,ds. \tag{76}$$

If we insert in (76) the expressions for Y and X given by (45), we find that M is given by [6]

[6] Although N is assumed to be zero in this discussion, we include it in eq (77) to show that a constant normal traction produces no twisting moment.

$$M=\int_C\{(-yN+xT)\cos(x,\nu)+(yT+xN)\cos(y,\nu)\}\,ds. \tag{77}$$

By making use of Green's theorem (52), we may rewrite (77) in the form

$$M=2T\int_R dS=2\pi abT. \tag{78}$$

We insert the expression (74) into (37) and (41) and, after some manipulation obtain

$$\phi(x,y)=\frac{1}{a^2n+b^2l}\{-b^2mx^2+(b^2l-a^2n)xy+a^2my^2\} \tag{79}$$

as the solution of the systems (37) and (41).

Using the value of $\phi(x,y)$ given by (79) in the defining equation (59) we find that D′ is given by

$$D'=\frac{\pi a^3b^3(nl-m^2)}{(b^2l+a^2n)}. \tag{80}$$

Hence, from (60), it follows that

$$\alpha=\frac{2(b^2l+a^2n)T}{a^2b^2(nl-m^2)}. \tag{81}$$

We next insert the expression (75) for $\psi(x,y)$ into the two equations (47) and after setting $x=a\cos\theta$, $y=b\sin\theta$, and making use of (73), we obtain a pair of polynomials in $\sin\theta$ and $\cos\theta$. These are expanded in Fourier series and the coefficients of the various terms are equated to zero. We thus find that, in order to satisfy the boundary conditions, $\psi(x,y)$ must be given by the expression

$$\begin{aligned}24a^2b^2(a^2n+b^2l)\cdot\psi(x,y)=&\,b^4(3lmb^2-5mna^2+\gamma)x^4+\\&4a^2b^2(b^2m^2-a^2n^2)x^3y+2a^2b^2\gamma x^2y^2\\&+4a^2b^2(b^2l^2-a^2m^2)xy^3+a^4(5lmb^2-3mna^2+\gamma)y^4-\\&2a^2b^4(3lmb^2-3mna^2+\gamma)x^2-\\&2a^4b^2(3lmb^2-3mna^2+\gamma)y^2,\end{aligned} \tag{82}$$

where γ is a constant, as yet arbitrary.

If the value of $\psi(x,y)$ given by (82) is inserted in the differential equation (39), it is found that γ must satisfy the relation

$$\begin{aligned}&\{3b_{22}b^4+2a^2b^2(b_{12}+b_{66})+3b_{11}a^4\}\cdot\gamma\\&=3b_{22}b^4(5mna^2-3lmb^2)+9b_{26}a^2b^2(b^2m^2-a^2n^2)+\\&\quad 9b_{16}a^2b^2(b^2l^2-a^2m^2)+3b_{11}a^4(3mna^2-5lmb^2)-\\&\quad 6b_{2}a^2b^4m+6b_{6}a^2b^2(a^2n-b^2l)+6b_{1}a^4b^2m+\\&\qquad 3a^2b^2(a^2n+b^2l)(b_{26}n-b_{16}l).\end{aligned} \tag{83}$$

Inserting the values of $\phi(x,y)$, α, and $\psi(x,y)$ given above into (36) we obtain the stress system

$$\left.\begin{aligned}\frac{\tau_{xx}}{T}&=\frac{(3mna^2-3lmb^2-\gamma)(b^2x^2+3a^2y^2-a^2b^2)-6a^2b^2(ln-m^2)xy}{3a^2b^4(ln-m^2)}\\[2ex]\frac{\tau_{yy}}{T}&=\frac{(3mna^2-3lmb^2-\gamma)(3b^2x^2+a^2y^2-a^2b^2)+6a^2b^2(ln-m^2)xy}{3a^4b^2(ln-m^2)}\\[2ex]\frac{\tau_{xy}}{T}&=\frac{3(ln-m^2)(b^2x^2-a^2y^2)-2(3mna^2-3lmb^2-\gamma)xy}{3a^2b^2(ln-m^2)}\\[2ex]\frac{\tau_{yz}}{T}&=\frac{4x(h-z)}{a^2}\\[2ex]\frac{\tau_{zx}}{T}&=-\frac{4y(h-z)}{b^2}\\[2ex]\frac{\tau_{zz}}{T}&=p'x^2+q'xy+r'y^2+A+Bx+Cy,\end{aligned}\right\} \tag{84}$$

where

$$\left.\begin{aligned}p'&=\frac{3b^2c_{22}(3lmb^2-5mna^2+\gamma)-3a^2c_{26}(b^2m^2-a^2n^2)+a^2c_{21}\gamma}{6a^2c_{23}(b^2l+a^2n)}-\frac{c_{31}m+c_{36}n}{c_{33}}+\frac{b^2m}{(b^2l+a^2n)}\\[2ex]q'&=\frac{3c_{32}(b^2m^2-a^2n^2)-c_{36}\gamma+3c_{21}(b^2l^2-a^2m^2)+3c_{23}(a^2n-b^2l)}{3c_{23}(b^2l+a^2n)}\\[2ex]r'&=\frac{b^2c_{32}\gamma-3b^2c_{26}(b^2l^2-a^2m^2)+3a^2c_{21}(5lmb^2-3mna^2+\gamma)}{6b^2c_{23}(b^2l+a^2n)}+\frac{c_{32}m+c_{26}l}{c_{23}}-\frac{a^2m}{b^2l+a^2n}.\end{aligned}\right\} \tag{85}$$

To evaluate the parameters A, B, C, in order that the conditions

$$\int_R \tau_{zz}dS=0, \qquad \int_R x\tau_{zz}dS=0, \qquad \int_R y\tau_{zz}dS=0$$

are satisfied we write

$$\left.\begin{aligned}
&\int_R (p'x^2+q'xy+r'y^2+A+Bx+Cy)\,dxdy=0\\
&\int_R (p'x^3+q'x^2y+r'xy^2+Ax+Bx^2+Cxy)\,dxdy=0\\
&\int_R (p'x^2y+q'xy^2+r'y^3+Ay+Bxy+Cy^2)\,dxdy=0.
\end{aligned}\right\} \tag{86}$$

If we denote by R, I_x, and I_y the area and the moments of inertia of the elliptic section, the system (86) can be written in the form

$$p'I_y+r'I_x+AR=0, \quad BI_y=0, \quad CI_x=0. \tag{87}$$

For the ellipse,

$$R=\pi ab, \quad I_x=\frac{\pi b^3 a}{4}, \quad I_y=\frac{\pi a^3 b}{4}. \tag{88}$$

Hence the parameters A, B, and C are given by

$$A=-\frac{a^2p'+b^2r'}{4}, \quad B=0, \quad C=0, \tag{89}$$

and the proposed problem has been completely solved.

For the isotropic elliptic cylinder the solution may be simplified by using the relations given in section 5.

The solution for this case is given by:

$$\left.\begin{aligned}
\phi(x,y)&=\frac{b^2-a^2}{a^2+b^2}xy\\
D'&=\frac{\pi a^3b^3}{2(a^2+b^2)}\\
\alpha&=\frac{4(a^2+b^2)T}{a^2b^2}\\
\gamma&=0\\
\psi(x,y)&=\frac{b^2xy^3-a^2x^3y}{12(a^2+b^2)}\\
\frac{\tau_{xz}}{T}&=-\frac{2xy}{b^2}\\
\frac{\tau_{yz}}{T}&=\frac{2xy}{a^2}\\
\frac{\tau_{xy}}{T}&=\frac{b^2x^2-a^2y^2}{a^2b^2}
\end{aligned}\right\} \tag{90}$$

$$\left.\begin{aligned}
\frac{\tau_{yz}}{T}&=\frac{4x(h-z)}{a^2}\\
\frac{\tau_{xz}}{T}&=-\frac{4y(h-z)}{b^2}\\
\frac{\tau_{zz}}{T}&=\frac{2\sigma(a^2-b^2)xy}{a^2b^2}.
\end{aligned}\right\} \tag{90}$$

Con.

For an isotropic circular cylinder of radius a the results are given by

$$\left.\begin{aligned}
\phi(x,y)&=0\\
D'&=\frac{\pi a^4}{4}\\
\alpha&=\frac{8T}{a^2}\\
\psi(x,y)&=\frac{xy^3-x^3y}{24}\\
\frac{\tau_{xz}}{T}&=\frac{2xy}{a^2}\\
\frac{\tau_{yz}}{T}&=\frac{2xy}{a^2}\\
\frac{\tau_{xy}}{T}&=\frac{x^2-y^2}{a^2}\\
\frac{\tau_{yz}}{T}&=\frac{4x(h-z)}{a^2}\\
\frac{\tau_{xz}}{T}&=-\frac{4y(h-z)}{a^2}\\
\frac{\tau_{zz}}{T}&=0.
\end{aligned}\right\} \tag{91}$$

7. Methods for obtaining solutions of the boundary-value problems

The determination of the stresses and displacements in the twisted cylinder has been made to depend upon the solution of the two linear partial differential equations,

$$l\frac{\partial^2\phi}{\partial x^2}+2m\frac{\partial^2\phi}{\partial x\partial y}+n\frac{\partial^2\phi}{\partial y^2}=0, \tag{92}$$

and

$$b_{22}\frac{\partial^4\psi}{\partial x^4}-3b_{26}\frac{\partial^4\psi}{\partial x^3\partial y}+2(b_{12}+b_{66})\frac{\partial^4\psi}{\partial x^2\partial y^2}-3b_{16}\frac{\partial^4\psi}{\partial x\partial y^3}+b_{11}\frac{\partial^4\psi}{\partial y^4}=b_2\frac{\partial^2\phi}{\partial x^2}-2b_6\frac{\partial^2\phi}{\partial x\partial y}+b_1\frac{\partial^2\phi}{\partial y^2}+b_0, \tag{93}$$

together with the appropriate boundary conditions, (41) and (47).

It was shown in section 4 that (92) can, by a simple change of variables, be transformed into the form

$$\nabla^2\phi(x,y)=0, \tag{94}$$

and the various methods of potential theory are available for the solution of this problem of Neumann. An elegant method of solution of the Neumann problem, due to Muschelišvili, is discussed in Sokolnikoff [8, p. 170].

The determination of the function $\psi(x,y)$ is considerably more difficult. An equation similar to (93) arises in the theory of anisotropic plates and a great deal of work has been done on the problem of solving this equation. A series of papers by Green and Taylor [2] gives a method of solution of the system (93)-(47) for the special case (corresponding to the orthotropic beam), where

$$b_{26}=b_{16}=0, \tag{95}$$

A perturbation method of solving the system (93)-(47), which consists in expressing the solution as a series of biharmonic functions, is described by Sokolnikoff [9]. Another, more elegant method, essentially due to Muschelišvili, depending on conformal mapping is also described by Sokolnikoff [7, p. 375] and [10].

If the beam is isotropic, (93) has the simple form

$$\nabla^4\psi(x,y)=0, \tag{96}$$

and the solution is easily obtained in the form

$$\psi(x,y)=\boldsymbol{R}[f(z)+\bar{z}g(z)], \tag{97}$$

where $f(z)$ and $g(z)$ are analytic functions of the complex variable $z[\equiv x+iy]$, which are chosen so that the function $\psi(x,y)$ given by eq (97) will satisfy the boundary conditions (47) [10].

8. Summary

The determination of the state of stress and the displacements in a built-in anisotropic cylinder twisted by forces uniformly applied along the lateral surface has been made to depend on the solution of two boundary-value problems.

The existence of the solutions of the boundary-value problems has been demonstrated, and several methods of solution of the boundary-value problems are indicated.

As an illustration of the method, the stresses in an anistropic beam of elliptic cross section twisted by constant tangential traction were obtained by assuming that the stress functions were second- and fourth-degree polynomials.

In a succeeding paper it is planned to extend the theory to the case where the applied surface tractions vary along the length of the cylinder, that is, where the boundary conditions are given in the form.[9]

$$\left.\begin{aligned} X&=f(z)X(s)\\ Y&=g(z)Y(s) \quad \text{on } C.\\ Z&=0. \end{aligned}\right\} \tag{98}$$

It is hoped that the theory can be further extended to the problems of extension, bending, and flexure of anisotropic cylindrical beams by forces distributed on the lateral surface.[10]

The boundary conditions for these problems are:

(a) Extension:

$$X=0 \quad \text{on } C$$

$$Y=0 \quad \text{on } C,$$

and Z is prescribed on C subject to the conditions

$$\int_C Z\,ds\neq 0, \quad \int_C xZ\,ds=0, \quad \int_C yZ\,ds=0. \tag{99}$$

(b) Bending:

$$X=0 \quad \text{on } C$$

$$Y=0 \quad \text{on C},$$

and Z is prescribed on C subject to the conditions

$$\int_C Z\,ds=0, \quad \int_C xZ\,ds\neq 0, \quad \int_C yZ\,ds\neq 0. \tag{100}$$

(c) Flexure:

$$Z=0 \quad \text{on } C,$$

and X and Y are prescribed on C subject to the conditions

$$\int_C X\,ds\neq 0, \quad \int_C Y\,ds\neq 0, \quad \int_C (xY-yX)\,ds=0. \tag{101}$$

It is apparent that a superposition of solutions corresponding to the boundary conditions (98) through (101) can be made to satisfy quite general boundary conditions on the lateral surface.

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[9] Zwolinsky and Riz [12] treated the case (for isotropic cylinders) where $f(z)\equiv z$, $N=0$, $T=$constant on C.

[10] The problems of extension, bending, and flexure of isotropic (and orthotropic) cylinders by forces applied on the end sections have been solved by Saint-Venant [8 p. 100].

9. References

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10. Appendix. Strain Energy and Elastic Symmetry

The strain energy, W, of a strained elastic medium is defined by [8, p. 88]

$$2W=\tau_{xx}e_{xx}+\tau_{yy}e_{yy}+\tau_{zz}e_{zz}+2\tau_{yz}e_{yz}+2\tau_{zx}e_{zx}+2\tau_{xy}e_{xy}. \quad (102)$$

In the linear theory of elasticity the strain energy is assumed to be given also by a positive definite quadratic form in the components of the strain tensor by

$$\left.\begin{aligned}W=&\tfrac{1}{2}b_{11}e_{xx}^2+b_{12}e_{xx}e_{yy}+b_{13}e_{xx}e_{zz}+b_{14}e_{xx}e_{yz}+b_{15}e_{xx}e_{zx}+b_{16}e_{xx}e_{xy}\\&+\tfrac{1}{2}b_{22}e_{yy}^2+b_{23}e_{yy}e_{zz}+b_{24}e_{yy}e_{yz}+b_{25}e_{yy}e_{zx}+b_{26}e_{yy}e_{xy}\\&+\tfrac{1}{2}b_{33}e_{zz}^2+b_{34}e_{zz}e_{yz}+b_{35}e_{zz}e_{zx}+b_{36}e_{zz}e_{xy}\\&+\tfrac{1}{2}b_{44}e_{yz}^2+b_{45}e_{yz}e_{zx}+b_{46}e_{yz}e_{xy}\\&+\tfrac{1}{2}b_{55}e_{zx}^2+b_{56}e_{zx}e_{xy}\\&+\tfrac{1}{2}b_{66}e_{xy}^2.\end{aligned}\right\} \quad (103)$$

The positive definite nature of the form (103) imposes certain requirements on the signs of the elements of the matrix

$$\begin{pmatrix}b_{11}&b_{12}&b_{13}&b_{14}&b_{15}&b_{16}\\b_{12}&b_{22}&b_{23}&b_{24}&b_{25}&b_{26}\\b_{13}&b_{23}&b_{33}&b_{34}&b_{35}&b_{36}\\b_{14}&b_{24}&b_{34}&b_{44}&b_{45}&b_{46}\\b_{15}&b_{25}&b_{35}&b_{45}&b_{55}&b_{56}\\b_{16}&b_{26}&b_{36}&b_{46}&b_{56}&b_{66}.\end{pmatrix} \quad (104)$$

An elastic medium is said to possess elastic symmetry in a plane if a reflection of coordinates in that plane leaves the strain energy form (103) invariant. Let us consider elastic symmetry in the xy-plane. The form (103) must be invariant under the transformation

$$\left.\begin{aligned}x'&=x\\y'&=y\\z'&=-z.\end{aligned}\right\} \quad (105)$$

The components of the strain tensor transform according to the relations

$$\left.\begin{aligned}&e_{x'x'}=e_{xx},\quad e_{y'y'}=e_{yy},\quad e_{z'z'}=e_{zz}\\&e_{y'z'}=-e_{yz},\quad e_{z'x'}=-e_{zx},\quad e_{x'y'}=e_{xy},\end{aligned}\right\} \quad (106)$$

and if the form (103) is to be invariant under the transformation, we must have

$$b_{14}=b_{15}=b_{24}=b_{25}=b_{35}=b_{46}=b_{56}=0. \quad (107)$$

The matrix (104) then has the form

$$\begin{pmatrix}b_{11}&b_{12}&b_{13}&0&0&b_{16}\\b_{12}&b_{22}&b_{23}&0&0&b_{26}\\b_{13}&b_{23}&b_{33}&0&0&b_{36}\\0&0&0&b_{44}&b_{45}&0\\0&0&0&b_{45}&b_{55}&0\\b_{16}&b_{26}&b_{36}&0&0&b_{66}.\end{pmatrix} \quad (104a)$$

Since the matrix (104a) is obtained from a positive definite quadratic form it follows that

$$\begin{vmatrix}b_{44}&b_{45}\\b_{45}&b_{55}\end{vmatrix}>0. \quad (108)$$

By differentiating (102) and (103) with respect to e_{xx}, and comparing, Hooke's law is obtained, expressing the stresses in terms of the strains.

If this expression of Hooke's law is solved for the strains in terms of stresses and the coefficients are denoted by $c_{11}, \ldots, c_{66}$ as in section 2, it will be seen that the determinant

$$\begin{vmatrix}c_{44}&c_{45}\\c_{45}&c_{55}\end{vmatrix} \quad (108a)$$

is a positive multiple of the determinant in (108), and hence is positive, as asserted in section 4.

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On Methods for Obtaining Solutions of Fixed End-Point Problems in the Calculus of Variations[1]

Marvin L. Stein[2]

Two methods for constructing solutions to the problem of minimizing an integral in a certain class of arcs joining a pair of fixed points are proposed. One of these procedures is a generalization of Newton's method, while the other is a "gradient" method. Conditions for convergence to a strong relative minimum are given in both cases.

1. Introduction

In this paper we consider methods for effectively constructing solutions to the problem of minimizing an integral

$$I(y)=\int_a^b f(x,y,y')\,dx \qquad (1)$$

in a certain class of arcs $y_i(x)$, $(a\leq x\leq b;\ i=1,2,\ldots,n)$ joining the two fixed points $(x=a,\ y(a)=\rho)$ $(x=b,\ y(b)=\sigma)$. As is well known, the minimum must necessarily satisfy the Euler equations

$$\frac{d}{dx}f_{y_i'}(x,y,y')=f_{y_i}(x,y,y') \qquad (i=1,2,\ldots,n).$$

Thus in any particular case the solution of these equations becomes of prime importance. The two methods to be proposed below enable us to compute arcs that satisfy the Euler equations subject to boundary conditions at two distinct points. For both methods an initial estimate of the solution is made. From this estimate a new estimate is obtained in a definite way, and so forth. These procedures have the advantage of giving estimates that satisfy the boundary conditions at each step, and their iterative nature makes them adaptable to automatic computation.

The first method of obtaining new estimates that will be discussed is a generalization of Newton's method for functions of a single real variable. This method is described in section 2 where it is actually applied to a more general class of systems of differential equations than the Euler equations, namely those systems of the form

$$\frac{d}{dx}g_i(x,y,y')=h_i(x,y,y') \qquad (i=1,2,\ldots,n).$$

To construct the convergence proof, we make the class of functions under consideration into a Banach space and regard the system of differential equations as an operator T on the space into itself. The necessary Banach space tools are developed in sections 3, 4, and 5. Sections 6, 7, and 8 are devoted to an analysis of the properties of T and of its first variation. The results obtained in these sections are used in section 9 to show that under simple conditions on the initial estimate the sequence of estimates derived from the first method will converge quadratically to an admissible function that satisfies the system of differential equations and the two-point boundary conditions. In fact, it is demonstrated in section 9 that convergence will occur if the initial estimate has nonconjugate end points (see definition 5) and is contained in some neighborhood on which the norm of T is bounded by a suitable constant. These conditions are shown to be satisfied if a solution exists and has nonconjugate end points. In this case there will exist a neighborhood of the solution such that the sequence of estimates will converge for any initial estimate chosen from this neighborhood. In the special case of the Euler equations, considered in section 11, it is shown that the limit function not only satisfies the differential equations but under the proper conditions also affords a strong relative minimum to $I(y)$. Section 10 concerns itself with an interesting lemma about conjugate points, that proves useful in section 11.

While not having as large a range of applicability as the first method, the second of the two methods of choosing new estimates that will be discussed has the advantage of being simpler to handle at each iteration. This method is described in section 12, where it is shown to be particularly useful for the solution of the Euler equations arising from an integral whose integrand is quadratic in y and y'. In this case the integral itself, instead of the differential equations, is considered; and it is shown that under suitable hypothesis the sequence of estimates converges to an arc that affords a strong relative minimum to $I(y)$. In fact, convergence will occur if I has a lower bound on the class of admissible functions and the second variation is positive.

Neither of the methods to be described below is presented as a quick computational procedure for a desk computer. However, they are processes that might well be carried out on automatic computing machinery. Numerical examples are to be found in section 13.

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[2] Present address, Consolidated-Vultee Aircraft Corporation, San Diego, California.

2. An Iterative Method for Solving Certain Systems of Second Order Differential Equations

In the following pages we shall be concerned with constructing solutions for systems of the form

$$\frac{d}{dx}(g_i(x,y,y'))=h_i(x,y,y') \qquad (i=1,2,\ldots,n)$$
$$y_i(a)=\rho_i, \quad y_i(b)=\sigma_i. \tag{2}$$

Here $y\equiv(y_1,y_2,\ldots,y_n)$ denotes an arc defined on the interval $a\leq x\leq b$ and ρ, σ denote arbitrary constant vectors. By formal integration of (2) it is found that an arc y that satisfies these equations must also satisfy

$$T_i(x;y)\equiv\int_a^x\left[g_i(t,y,y')-\int_a^t h_i(s,y,y')ds\right]dt$$
$$-\frac{x-a}{b-a}\int_a^b\left[g_i(t,y,y')-\int_a^t h_i(s,y,y')ds\right]dt=0$$
$$y_i(a)=\rho_i, \quad y_i(b)=\sigma_i \qquad (i=1,2,\ldots,n). \tag{3}$$

Conversely, any arc without corners that satisfies (3) must also satisfy (2). For reasons that will be clarified below, we prefer to make use of the differential eq (2) in their integral form (3). Accordingly, let us examine some of the properties of the function $T_i(x;y)$.

That

$$T_i(a;y)=T_i(b;y)=0 \qquad (i=1,2,\ldots,n) \tag{4}$$

independent of the arc y is immediately apparent.

One recalls that a function of one or more variables is said to be of *class* C^n, $n\geq 0$, in its arguments on its domain of definition if it is continuous on that domain together with all of its (partial) derivatives up to and including those of order n. An arc $y\equiv(y_1,y_2,\ldots,y_n)$ is of class C^n if all of its components, y_i, have this property. From further inspection of the definition of $T_i(x;y)$ it is evident that if g and h are of class C^{n-1} and y is of class C^k for some k on $0\leq k\leq n$, then $T_i(x;y)$, considered as a function of x, is also of class C^k that is, $T(x;y)$ belongs to the same class of arcs as y, provided g and h have proper differentiability properties. Let a primed variable signify differentiation of that variable with respect to x. Then it will be noted that the arc $T''(x;y)$, which when set equal to zero merely presents the differential eq (2) in their usual form, does not possess the property of belonging to the same class to which y belongs. In fact, even if y is of class C^n, $T''(x;y)$ may only belong to class C'.

Before describing an iterative procedure for constructing arcs that satisfy (3), one must first introduce the functions

$$\delta g_i(x,y;\eta)\equiv g_{iy_j}(x,y,y')\eta_j+g_{iy_j'}(x,y,y')\eta_j'$$
$$\delta h_i(x,y;\eta)\equiv h_{iy_j}\eta_j+h_{iy_j'}\eta_j' \qquad (i,j=1,2,\ldots,n) \tag{5}$$

and

$$\delta T_i(x,y;\eta)\equiv\int_a^x\left[\delta g_i(t,y;\eta)-\int_a^t\delta h_i(s,y;\eta)ds\right]dt$$
$$-\frac{x-a}{b-a}\int_a^b\left[\delta g_i-\int_a^t\delta h_i ds\right]dt \qquad (i=1,2,\ldots,n). \tag{6}$$

In (5), (6), and elsewhere in this paper, unless otherwise noted, the tensor convention of summing on repeated indices is utilized. The arc η is taken to be such that it vanishes at the end points of $a\leq x\leq b$.

The function $\delta T_i(x,y;\eta)$ can be obtained from $T_i(x;y)$ as

$$\delta T_i(x,y;\eta)=\frac{d}{d\alpha}T_i(x;y+\alpha\eta)|_{\alpha=0}$$

and is often referred to as the "variation" of T_i. It has properties similar to those of T_i. For example, δT_i, considered as a function of x belongs to the same class as the function η, provided that g, h, and y have sufficient differentiability. This fact follows from a small bit of reflection on (6) as does the fact that

$$\delta T_i(a,y;\eta)=\delta T_i(b,y;\eta)=0 \qquad (i=1,2,\ldots,n), \tag{7}$$

regardless of the choice of y and η. It is seen that these properties do not hold for the variation of the differential equations (2) taken in their usual form.

In the iteration process to be discussed here, it is assumed that the i^{th} estimate $y^{(i)}$ of an arc satisfying (3) has been obtained. An "improved estimate" $y^{(i+1)}$ is then found as follows. Determine a "variation" η by solving the system

$$T_j(x;y^{(i)})+\delta T_j(x,y^{(i)};\eta)=0 \qquad (j=1,2,\ldots,n)$$
$$\eta_j(a)=\eta_j(b)=0. \tag{8}$$

The function $y^{(i+1)}$ is then chosen to be

$$y^{(i+1)}=y^{(i)}+\eta^{(i)}, \tag{9}$$

$y^{(i)}$ is replaced by $y^{(i+1)}$, and the steps just described are repeated.

This method may be justified heuristically by observing that upon expanding $T_i(x;y+\eta)$ by Taylor's theorem one gets

$$T_i(x;y+\eta)=T_i(x;y)+\delta T_i(x,y;\eta)+\ldots$$

Thus it is reasonable to assume that a good approximation to a solution of

$$T_i(x;y+\eta)=0 \qquad (i=1,2,\ldots,n)$$

for a known function y might be obtained by solving

$$T_i(x;y)+\delta T_i(x,y;\eta)=0 \qquad (i=1,2,\ldots,n) \quad (10)$$

for η. The analogy between this procedure and Newton's method for a function of a single real variable is clear. Consequently, it will henceforth be referred to as Newton's method.

As one sees by setting $g_i(x,y,y')\equiv y_i'(i=1,2,\ldots,n)$, the differential equations (2) contain the important class of equations

$$y_i''=h_i(x,y,y') \qquad (i=1,2,\ldots,n) \qquad (11)$$

as a special case. By setting

$$\begin{aligned} g_i(x,y,y')&\equiv f_{y_i'}(x,y,y') \qquad (i=1,2,\ldots,n)\\ h_i(x,y,y')&\equiv f_{y_i}(x,y,y') \qquad (i=1,2,\ldots,n) \end{aligned} \quad (12)$$

(2) can also be made to assume the form

$$\begin{aligned} &\frac{d}{dx}f_{y_i'}=f_{y_i} \qquad (i=1,2,\ldots,n)\\ &y_i(a)=\rho_i, \qquad y_i(b)=\sigma_i. \end{aligned} \quad (13)$$

These are the Euler equations associated with the integral (1), and their solution is the chief application of Newton's method which we wish to make in this paper. The functions T_i and δT_i associated with (13) are, respectively,

$$T_i(x;y)=\int_a^x\left[f_{y_i'}-\int_a^t f_{y_i}ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[f_{y_i'}-\int_a^t f_{y_i}ds\right]dt$$

and

$$\delta T_i(x,y;\eta)=\int_a^x\left[\omega_{\eta_i'}-\int_a^t\omega_{\eta_i}ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[\omega_{\eta_i'}-\int_a^t\omega_{\eta_i}ds\right]dt,$$

where $\omega_{\eta_i'}$ and ω_{η_i} are derivatives of the quadratic form

$$2\omega(x,y;\eta)=f_{y_iy_j}(x,y,y')\eta_i\eta_j+2f_{y_iy_j'}\eta_i\eta_j'+f_{y_i'y_j'}\eta_i'\eta_j' \qquad (i,j=1,2,\ldots,n).$$

A deeper insight into Newton's method as applied to (13) may be acquired by applying Taylor's theorem to the integrand $f(x;y+\eta,y'+\eta')$. The expansion results in

$$\begin{aligned} f(x,y+\eta,y'+\eta')&=f(x,y,y')+f_{y_i}(x,y,y')\eta_i+f_{y_i'}(x,y,y')\eta_i'\\ &\quad+\tfrac{1}{2}f_{y_iy_j}\eta_i\eta_j+f_{y_iy_j'}\eta_i\eta_j'+\tfrac{1}{2}f_{y_i'y_j'}\eta_i'\eta_j'+\ldots\\ &=f(x,y,y')+f_{y_i}(x,y,y')\eta_i+f_{y_i'}\eta_i'+\omega(x,y,\eta)+\ldots. \end{aligned}$$

Let us drop the higher order terms and consider the problem of minimizing

$$J(y+\eta)=\int_a^b[f(x,y,y')+f_{y_i}\eta_i+f_{y_i'}\eta_i'+\omega(x,y;\eta)]dx,$$

where y is a known function and η is variable. The Euler equations with respect to η of this integral are

$$\frac{d}{dx}f_{y_i'}-f_{y_i}+\frac{d}{dx}\omega_{\eta_i'}-\omega_{\eta_i}=0 \qquad (i=1,2,\ldots,n).$$

This last system of differential equations is formally equivalent to (10). Thus Newton's method consists of repeatedly replacing $f(x,y,y')$ with an integrand that is quadratic and minimizing the new integral so obtained. Consequently, if the original integrand is quadratic, Newton's method should yield a solution in a single step independently of the choice of the initial estimate. That this actually occurs is readily verifiable.

Finally, we note that although the Euler equations often have an equivalent formulation as a system of type (11), Newton's method may not necessarily be the same for both forms of the problem. This is due to the fact that in general δT is not invariant under a transformation of T to an equivalent form. For example, the two differential equations

$$\frac{d}{dx}(yy')=1+2y'^2,$$

and

$$y''=\frac{1+y'^2}{y}$$

are equivalent for $y>0$. In the first case

$$T(x,y)=\tfrac{1}{2}[y^2(x)-y^2(a)]-\int_a^x\left[\int_a^t(1+2y'^2)ds\right]dt-\frac{x-a}{b-a}\left\{\tfrac{1}{2}[y^2(b)-y^2(a)]-\int_a^b\left[\int_a^t(1+2y'^2)ds\right]dt\right\},$$

and

$$\delta T(x,y;\eta)=y(x)\eta(x)-4\int_a^x\left[\int_a^t y'\eta'ds\right]dt+\frac{x-a}{b-a}\int_a^b\left[\int_a^t y'\eta'ds\right]dt.$$

In the second case

$$T(x;y)=y(x)-y(a)-\int_a^x\left[\int_a^t\left(\frac{1+y'^2}{y}\right)ds\right]dt-\frac{x-a}{b-a}\left\{y(b)-y(a)-\int_a^b\left[\int_a^t\left(\frac{1+y'^2}{y}\right)ds\right]dt\right\},$$

and

$$\delta T(x,y;\eta)=\eta(x)-\int_a^x\left[\int_a^t\left(\frac{2y'\eta'}{y}-\frac{1+y'^2}{y^2}\cdot\eta\right)ds\right]dt$$

$$-\frac{x-a}{b-a}\int_a^b\left[\int_a^t\left(\frac{2y'}{y}\cdot\eta'-\frac{1+y'^2}{y^2}\cdot\eta\right)ds\right]dt.$$

It is seen that the two variations are not the same. Since the convergence theory to be developed below will be general enough to include either (11) or (13), one will be free to use whichever system of differential equations seems to be simpler in any given problem.

3. Preliminaries

There are several definitions and known results of general analysis to which there will be frequent occasion to refer. It will be convenient to have these collected in one section. Accordingly, the next few pages are devoted to this end. The reader who wishes more detail about what is to follow in this section is advised to consult a paper of Graves and Hildebrandt [1].[2] The treatment he will find there is more general and couched in somewhat different terminology.

Functions defined on a linear subspace U of a Banach space Y into itself are dealt with here. These functions may depend, in addition, upon a parameter chosen from Y or a region of Y. When the arguments are indicated, the parameter will always appear first and be denoted by a lower case y perhaps bearing a superscript. It will be set off from the independent variables, which will be denoted by Greek letters η, ζ, ξ etc., by a semicolon.

DEFINITION 1. $F(y;\eta)$ is said to be *continuous in y at $y^{(0)}$ uniformly relative to η* if given $\epsilon>0$ there exists $\delta>0$ such that

$$||F(y;\eta)-F(y^{(0)};\eta)||<\epsilon||\eta|| \tag{14}$$

for all y in $(y^{(0)})_\delta$ and every η in U. Here and elsewhere $(y^{(0)})_\delta$ and similar notations denote the δ-neighborhood of $y^{(0)}$.

The extension of definition 1 to the case of a function $F(y;\eta,\zeta,\ldots,\xi)$ of several independent variables is clear. The only essential change is made in replacing eq (14) by

$$||F(y;\eta,\zeta,\ldots,\xi)-F(y^{(0)};\eta,\zeta,\ldots,\xi)||$$
$$<\epsilon||\eta||\cdot||\zeta||\ldots||\xi||.$$

All of the definitions that follow may be generalized to functions of several variables in a similar way.

DEFINITION 2. Let Y_0 be a region of Y. $F(y;\eta)$ is said to be *uniformly continuous in y on Y_0 uniformly with respect to η* if for every $\epsilon>0$ there exists a $\delta>0$ such that

$$||F(y^{(1)};\eta)-F(y^{(2)};\eta)||<\epsilon||\eta||$$

for all $y^{(1)}$, $y^{(2)}$ in Y_0 that satisfy $||y^{(1)}-y^{(2)}||\leq\delta$ and all η in U.

DEFINITION 3. $F(y^{(0)};\eta)$ is said to be *bounded at $y^{(0)}$ uniformly with respect to η* if there exists a constant $M\geq0$ such that

$$||F(y^{(0)};\eta)||\leq M||\eta||$$

for all $\eta\epsilon U$. If some $M\geq0$ holds for all y belonging to some neighborhood $(y^{(0)})_a$, F is said to be *uniformly bounded on $(y^{(0)})_a$ uniformly with respect to η*.

For the sake of brevity the words "uniformly with respect to η", that appear in the above definitions and that should appear in the concluding definition (4) of this section, will henceforth be omitted. The reader, however, should supply them for himself in all of the necessary places.

The following lemma is an easy consequence of the definitions.

LEMMA 1. *If $F(y;\eta)$ to U is continuous in y at $y^{(0)}$ and bounded at $y^{(0)}$, there exists a neighborhood of $y^{(0)}$ such that F is uniformly bounded on this neighborhood.*

The conclusion follows from the inequality

$$||F(y;\eta)||\leq||F(y^{(0)};\eta)||+||F(y;\eta)-F(y^{(0)};\eta)||.$$

DEFINITION 4. $F(y;\eta)$ to U is said to be of *class C' in y uniformly on Y_0* in case there exists a function $dF(y;\eta,\zeta)$ on Y_0UU to U having the following properties:

(1) dF is uniformly continuous in y on Y_0.
(2) dF is linear in ζ and uniformly bounded on Y_0.
(3) the function $R(y^{(1)};y^{(2)};\eta)$ defined by

$$F(y^{(2)};\eta)-F(y^{(1)};\eta)-dF(y^{(1)};\eta,y^{(2)}-y^{(1)})$$
$$=R(y^{(1)};y^{(2)};\eta)||y^{(2)}-y^{(1)}||,\quad y^{(2)}\neq y^{(1)},$$
$$R(y^{(1)};y^{(1)};\eta)=\theta \tag{15}$$

is such that given $\epsilon>0$ there exists $\delta>0$ such that

$$||R(y^{(1)};y^{(2)};\eta)||<\epsilon||\eta||$$

for all $y^{(1)},y^{(2)}$ in Y_0 satisfying $||y^{(2)}-y^{(1)}||\leq\delta$ and all η in U.

Here and elsewhere θ refers to the zero of the space Y.

LEMMA 2. *If F is of class C' in y uniformly on Y_0, there exist nonnegative constants α and M such that for every $y^{(1)}$ and $y^{(2)}$ of Y_0 for which $||y^{(1)}-y^{(2)}||\leq\alpha$ it is true that*

$$||F(y^{(1)};\eta)-F(y^{(2)};\eta)||\leq M||y^{(1)}-y^{(2)}||\cdot||\eta||$$

for all η in U.

The conclusion of this lemma follows from an application of (2) and (3) of definition 4 to eq (15).

Finally, we state a lemma that corresponds to lemma 16.3 of Graves [1].

LEMMA 3. *Let $F(y;\eta)$ to U be bounded at y and linear in η for each y of the region Y_0. In addition let $y^{(0)}$ be a point of Y_0 and let F have a bounded inverse*

[2] Figures in brackets indicate the literature references at the end of this paper.

$F^{-1}(y^{(0)};\eta)$ with respect to η, and be of class C' in y uniformly on Y_0. Then there exists a constant α such that for each y in $(y^{(0)})_\alpha$, $F^{-1}(y;\eta)$ exists and is of class C' in y uniformly on $(y^{(0)})_\alpha$.

4. A Convergence Theorem

In section 2 an iterative method for constructing solutions of the Euler equations was described. However, no mention was made of conditions of convergence. In this section we will describe and give conditions for the convergence of a similar procedure that can be applied to a large class of functions. Later the process of section 2 will be considered as a special case of that to be given here, and it will be shown that under simple assumptions the special process satisfies the hypothesis of the convergence theorem of this section.

Let $S(y)$ be a continuous function not identically zero defined on a region Y_0 of a Banach space Y to a subspace U. The problem of interest is that of obtaining solutions of the equation

$$S(y)=\theta. \tag{16}$$

Toward this end one introduces a linear operator $K(y;\eta)$ on U to U, which depends in addition on some parameter taken from Y. It is assumed that $K(y;\eta)$ is nonsingular in η. Let the ith estimate $y^{(i)}$ of the solution to (16) be known. The $i+1$st estimate is constructed as follows. A function $\eta^{(i)}$ is determined by solving

$$S(y^{(i)})+K(y^{(i)};\eta)=\theta \tag{17}$$

and the function $y^{(i+1)}$ is chosen to be

$$y^{(i+1)}=y^{(i)}+\eta^{(i)}. \tag{18}$$

The process is then repeated.

In order to simplify the statement of the convergence theorem, several conditions will now be introduced on the function $S(y)$ and on $K^{-1}(y;\eta)$, the inverse with respect to η of $K(y;\eta)$.

(I). *The arc $y^{(0)}$ of Y_0, the constant $\gamma>0$, and the operator $K^{-1}(y;\eta)$ are such that there exists a positive constant M for which*

$$||K^{-1}(y^{(1)};\eta)-K^{-1}(y^{(2)};\eta)||\leq M||y^{(1)}-y^{(2)}||\cdot||\eta|| \tag{19}$$

for all η in U and all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\gamma$. Furthermore, there exists $B\geq 0$ such that

$$||K^{-1}(y;\eta)||\leq B||\eta|| \tag{20}$$

for all y in $(y^{(0)})_\gamma$ and all η in U.

From (19) it is clear that K^{-1} is continuous in y at $y^{(0)}$. Therefore by lemma 1 one sees that in order for (20) to hold on some neighborhood of $y^{(0)}$, which is also a subneighborhood of $(y^{(0)})_\gamma$, it is sufficient to assume that $K^{-1}(y;\eta)$ is bounded at $y^{(0)}$. The remaining part of (I) is, of course, undisturbed on a subneighborhood of $(y^{(0)})_\gamma$.

In the next two conditions certain non-negative constants H, L, and N appear. These are taken to be such that they satisfy

$$H=B\cdot L+M\cdot N<1. \tag{21}$$

(II). *For all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\gamma$ the inequality*

$$||K(y^{(1)};y^{(1)}-y^{(2)})-S(y^{(1)})+S(y^{(2)})||\leq L||y^{(1)}-y^{(2)}|| \tag{22}$$

is true.

This condition asserts that K is an approximate differential of S on the neighborhood $(y^{(0)})_\gamma$.

(III). *There exists a constant C such that $0<C<\gamma$ and*

$$||S(y^{(0)})||\leq(1-H)C/B, \tag{23}$$

while

$$||S(y)||\leq N \tag{24}$$

for all other y in $(y^{(0)})_\gamma$.

THEOREM 1. *Let the initial estimate $y^{(0)}$ and the positive constant γ be such that (I), (II), (III) are valid. Then the sequence of estimates determined by (18) converges to a unique element $\overline{y}$ of $(y^{(0)})_\gamma$. The arc $\overline{y}$ is the unique solution of eq(16) in $(y^{(0)})_\gamma$.*

First we prove that

$$||y^{(i)}-y^{(i-1)}||\leq H^{i-1}(1-H)C,\ y^{(i)}\ in\ (y^{(0)})_\gamma;\qquad i\geq 1. \tag{25}$$

In order to construct the proof by induction consider the inequality

$$\begin{aligned}||y^{(1)}-y^{(0)}||&=||y^{(0)}+K^{-1}(y^{(0)};-S(y^{(0)}))-y^{(0)}||\\&\leq B||S(y^{(0)})||\leq(1-H)C<\gamma.\end{aligned}$$

This last follows from (23) and establishes (25) for $i=1$. Assume that (25) is true for all $i\leq m-1$. By this assumption in conjunction with (19), (22), and (24) one may write

$$\begin{aligned}||y^{(m)}-y^{(m-1)}||&=||y^{(m-1)}+K^{-1}(y^{(m-1)};-S(y^{(m-1)}))\\&\quad-y^{(m-2)}-K^{-1}(y^{(m-2)};-S(y^{(m-2)}))||\\&\leq||y^{(m-1)}-y^{(m-2)}-K^{-1}(y^{(m-1)};S(y^{(m-1)}))\\&\quad+K^{-1}(y^{(m-1)};S(y^{(m-2)}))||+||K^{-1}(y^{(m-2)};\\&\quad S(y^{(m-2)}))-K^{-1}(y^{(m-1)};S(y^{(m-2)}))||\\&\leq||K^{-1}(y^{(m-1)};K(y^{(m-1)};\\&\quad y^{(m-1)}-y^{(m-2)})-S(y^{(m-1)})+S(y^{(m-2)}))||\\&\quad+M||y^{(m-1)}-y^{(m-2)}||\cdot||S(y^{(m-2)})||\\&\leq B||K(y^{(m-1)};y^{(m-1)}-y^{(m-2)})-S(y^{(m-1)})+S(y^{(m-2)})||\\&\quad+M||y^{(m-1)}-y^{(m-2)}||.||S(y^{(m-2)})||\\&\leq(B\cdot L+M\cdot N)||y^{(m-1)}-y^{(m-2)}||\\&\leq H\cdot H^{m-2}(1-H)C.\end{aligned}$$

This establishes the first assertion of (25) for $i=m$. The rest of the proof of (25) follows from the inequality

$$||y^{(m)}-y^{(0)}||=||\sum_{i=1}^{m}(y^{(i)}-y^{(i-1)})||\leq\sum_{i=1}^{m}||y^{(i)}-y^{(i-1)}||$$

$$\leq\sum_{i=1}^{m}H^{i-1}(1-H)C=(1-H^{m})C<\gamma. \tag{26}$$

If a sequence $\{y^{(i)}\}$ in a Banach space is such that the series $\Sigma||y^{(i)}-y^{(i-1)}||$ is convergent, it has a unique limit. Therefore, by (25) there exists a unique element $\overline{y}$ of Y that satisfies

$$\lim_{i\to\infty}||y^{(i)}-\overline{y}||=0. \tag{27}$$

It follows from (26) and (27), that $\overline{y}$ must belong to $(y^{(0)})_\gamma$.

Upon passing to the limit in the expression

$$y^{(m)}=y^{(m-1)}-K^{-1}(y^{(m-1)};S(y^{(m-1)})),$$

and recalling the fact that K^{-1} and S are continuous, one obtains

$$K^{-1}(\overline{y};S(\overline{y}))=\theta.$$

This implies $S(\overline{y})=\theta$.

To see that $\overline{y}$ is the unique solution of (16) in $(y^{(0)})_\gamma$ let y^* be another element of this neighborhood satisfying (16). Then by (20) and (22)

$$||\overline{y}-y^*||=||\overline{y}-y^*-K^{-1}(\overline{y};S(\overline{y}))+K^{-1}(\overline{y};S(y^*))||$$

$$=||K^{-1}(\overline{y};K(\overline{y};\overline{y}-y^*)-S(\overline{y})+S(y^*))||$$

$$\leq B||K(\overline{y};\overline{y}-y^*)-S(\overline{y})+S(y^*)||$$

$$\leq BL||\overline{y}-y^*||.$$

Since (21) implies that $BL<1$, $\overline{y}\equiv y^*$.

It is possible to modify the method of obtaining iterated estimates by always holding y in $K(y;\eta)$ fixed at the initial estimate $y^{(0)}$. This is equivalent to originally introducing an operator $K(\eta)$, which depends only on functions from U. In this case, of course, the convergence theorem may be simplified. Condition (I) clearly no longer applies, and condition (III) may be modified so that (24) is eliminated. The resulting theorem may be stated in the following way.

THEOREM 2. *Let $K(\eta)$ be a nonsingular linear operator. Let $K^{-1}(\eta)$ be bounded and let the positive constant L be such that $||K^{-1}||L<1$. Then if the initial estimate $y^{(0)}$ and the positive constant α are such that*

(1) *for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\alpha$*

$$||K(y^{(1)}-y^{(2)})-S(y^{(1)})+S(y^{(2)})||\leq L||y^{(1)}-y^{(2)}||$$

(2) *for some C, $0<C<\alpha$*

$$||S(y^{(0)})||<\frac{(1-||K^{-1}||\cdot L)C}{||K^{-1}||}$$

the sequence of estimates $\{y^{(i)}\}$ will converge to a unique element $\overline{y}$ of $(y^{(0)})_\alpha$. The arc $\overline{y}$ will satisfy eq (16) uniquely in $(y^{(0)})_\alpha$.

Thus when K is independent of y, only an initial approximation to the solution at one point and an approximate differential of S are required to insure convergence. Consequently, this last theorem is seen to be just the classical implicit function theorem in general analysis. One notes that in this case the continuity of S is not required.

This section will be completed by establishing conditions under which the sequence of estimates will converge quadratically.

(IV). *Let $y^{(0)}$ be an element of Y and let γ and L' be constants such that*

$$||K(y^{(1)};y^{(1)}-y^{(2)})-S(y^{(1)})+S(y^{(2)})||$$

$$\leq L'||y^{(1)}-y^{(2)}||^2 \tag{28}$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\gamma$.

We may now state the following theorem.

THEOREM 3. *Let the initial estimate $y^{(0)}$ and the positive constant γ be such that (I), (III), and (IV) are valid. Then the sequence of estimates determined by (18) converges quadratically to a unique element $\overline{y}$ of some neighborhood $(y^{(0)})_\alpha$, $\alpha\leq\gamma$. The arc $\overline{y}$ is the unique solution of eq (16) in $(y^{(0)})_\alpha$.*

To see that (IV) implies (II) on $(y^{(0)})_\alpha$, $\alpha\leq\gamma$, merely take α small enough so that $L'||y^{(1)}-y^{(2)}||\leq L$ for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\alpha$. As (I) and (III) are undisturbed on the smaller neighborhood, all conclusions but that of quadratic convergence follow at once from theorem 1. The quadratic nature of the convergence can be established by the following calculations. From (17) and (18) it follows that

$$y^{(i)}-y^{(i-1)}=K^{-1}(y^{(i-1)};-S(y^{(i-1)})). \tag{29}$$

Therefore,

$$K(y^{(i-1)};y^{(i)}-y^{(i-1)})=-S(y^{(i-1)}).$$

Thus by (28)

$$||S(y^{(i)})||\leq L'||y^{(i)}-y^{(i-1)}||^2.$$

Making use of this last result in (29) one obtains

$$||y^{(i)}-y^{(i-1)}||\leq BL'||y^{(i-1)}-y^{(i-2)}||^2.$$

This establishes theorem 3.

5. Some Properties of $\delta T(x,y;\eta)$

In this section some properties of the function δT, which was defined in section 2, are developed with an eye toward applying the considerations of

section 4 to Newton's method. First, however, the space Y to which the arcs y belong must be more precisely defined. The space Y is chosen to be the collection $(y_i(x))$ of all n-tuples of real valued functions that are of class C' on the interval $a \leq x \leq b$. Addition of elements in Y and multiplication of elements of Y by real scalars are defined in the usual manner. Under these definitions of addition and multiplication Y forms a linear vector space over the real numbers.

For any y in Y let

$$|y(x)| = \left(\sum_{i=1}^{n} |y_i(x)|^2 \right)^{\frac{1}{2}}.$$

Then a topology may be introduced into Y by defining a norm

$$||y|| = \sup_{a \leq x \leq b} |y(x)| \vee \sup_{a \leq x \leq b} |y'(x)|. \tag{30}$$

The fact that (30) satisfies the properties of a norm is easily verified as is the fact that $||y^{(1)} - y^{(2)}||$ has all of the properties of a metric. It is also not difficult to establish that any Cauchy sequence of elements of Y must converge in the sense of the norm to an element of Y. In fact, convergence in Y implies uniform convergence in the ordinary sense of the sequence of first derivatives. Thus in the topology specified by (30) Y is a Banach space.

The subclass of arcs in Y that have the property

$$y(a) = y(b) = 0 \tag{31}$$

will be denoted by the symbol U. It is clear that U is a linear subspace of Y.

In order to eliminate repetition when discussing certain properties of the norms of elements of Y, it is convenient to have the following simple lemma available for reference.

LEMMA 4. *Let y, η, and $Q(x,y;\eta)$ belong to Y. Let there correspond to each component Q_i of Q a non-negative number $C^{(i)}$ such that*

$$\sup_x |Q_i(x,y;\eta)| \leq C^{(i)} ||\eta||.$$

Also let there correspond to each component Q_i' of Q', the first derivative with respect to x of Q, a non-negative number $C^{(i)\prime}$ such that

$$\sup_x |Q_i'(x,y;\eta)| \leq C^{(i)\prime} ||\eta||.$$

Then there exists $C \geq 0$ such that

$$||Q(x,y;\eta)|| \leq C||\eta||.$$

By (30) either

$$||Q(x,y;\eta)|| = \sup_x \left(\sum_{i=1}^{n} |Q_i|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \sup_x |Q_i| \leq \sum_{i=1}^{n} C^{(i)} ||\eta||,$$

or

$$||Q(x,y;\eta)|| = \sup_x \left(\sum_{i=1}^{n} |Q_i'|^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^{n} \sup_x |Q_i'| \leq \sum_{i=1}^{n} C^{(i)\prime} ||\eta||.$$

Obviously C can be chosen to be the greater of the numbers $\sum_{i=1}^{n} C^{(i)}$ and $\sum_{i=1}^{n} C^{(i)\prime}$.

In addition to defining the space of functions under consideration, it is desirable at this point to state some assumptions that will hold throughout the remainder of the paper. Unless otherwise noted, it is assumed that the functions $g(x,y,y')$ and $h(x,y,y')$ are of class C^2 in a region R of Euclidean $2n+1$-space. Any arc of Y, all of whose elements $(x,y(x),y'(x))$ are in R, will be termed *admissible*. At least one admissible arc is required to exist. This implies the existence of a subregion Y_0, all of whose elements are admissible. It is also assumed that the determinant $|g_{iy_j'}(x,y,y')|$ never vanishes in R. In the case of the Euler equations, that is, when g and h are given by (12), the above assumptions imply that the derivatives f_{y_i} and $f_{y_i'}$ of the integrand $f(x,y,y')$ must be of class C_2 in R.

In eq (6), which defines the function $\delta T_i(x,y;\eta)$, let the parameter y range over Y_0 and the variable η range over U. The following result is then valid.

LEMMA 5. *For each y belonging to Y_0 the transformation $\delta T(x,y;\eta) \equiv (\delta T_1, \ldots, \delta T_n)$ is on $Y_0 U$ to U and is linear in η and bounded at y.*

The fact that δT is an element of U follows from the considerations of section 2. The linearity in η is a consequence of the corresponding property of eq (5) and of the operation of integration.

To simplify the proof of the boundedness we introduce a lemma that will be found useful in what follows.

LEMMA 6. *Let $q(x;y)$ be given by*

$$q(x;y) \equiv a_j(x) y_j(x) + b_j(x) y_j'(x),$$

where $a_j(x)$, $b_j(x)$ $(j = 1, 2, \ldots, n)$ are continuous on $a \leq x \leq b$ and y is in Y. Then there exists a constant $C \geq 0$ such that for each y in Y

$$\sup_x |q(x;y)| \leq C||y|| \tag{32}$$

and

$$\sup_x \left| \int_a^x q(t;y)\,dt \right| \leq (b-a)C||y||. \tag{33}$$

On recalling that $\sup_x |y_i| \leq ||y||$ and $\sup_x |y_i'| \leq ||y||$ for all $(i = 1, 2, \ldots, n)$ one quickly obtains the inequality

$$\sup_x |q(x;y)| \leq \left(\sum_j \sup_x |a_j| \right) ||y|| + \left(\sum_j \sup_x |b_j| \right) ||y|| = C||y||.$$

The second statement follows from the first by means of the inequality

$$\sup_x \left| \int_a^x q(t;y)\,dt \le \int_a^b |q(t;y)|\,dt \le \int_a^b \sup_t |q(t;y)|\,dt.$$

Upon noting that δg_i and δh_i are of the same type as $q(x;y)$ and applying lemma (6) one obtains the inequality

$$\begin{aligned}
|\delta T_i| &\le \sup_x |\delta T_i| \\
&= \sup_x \left| \int_a^x \left[\delta g_i - \int_a^t \delta h_i\,ds \right] dt - \right. \\
&\qquad \left. \frac{x-a}{b-a} \int_a^b \left[\delta g_i - \int_a^t \delta h_i\,ds \right] dt \right| \\
&\le 2 \int_a^b \left| \delta g_i - \int_a^t \delta h_i\,ds \right| dt \\
&\le 2 \int_a^b \left[\sup_x |\delta g_i| + \sup_t \left| \int_a^t \delta h_i\,ds \right| \right] dt \\
&\le C^{(i)} ||\eta||, \quad C^{(i)} \ge 0.
\end{aligned}$$

A similar result follows for $|\delta T_i'|$. Thus the conclusion follows from lemma 4.

6. An Existence Lemma

In section 7 the existence of an arc η in U that satisfies

$$\begin{aligned}
\delta T_i(x,y;\eta) \equiv & \int_a^x \left[\delta g_i - \int_a^t \delta h_i\,ds \right] dt - \\
& \frac{x-a}{b-a} \int_a^b \left[\delta g_i - \int_a^t \delta h_i\,ds \right] dt \\
= & l_i(x), \quad (i=1,2,\ldots,n) \qquad (34)
\end{aligned}$$

for an arbitrary function $l(x)$ in U will be discussed. To facilitate this discussion it is convenient to have available the following lemma.

LEMMA 7. *For each admissible arc y and every n-tuple of functions $m_i(x)$ continuous on $a \le x \le b$ the system*

$$\delta g_i(x,y;\eta) - \int_a^x \delta h_i(t,y;\eta)\,dt = m_i(x) \qquad (i=1,2,\ldots,n) \qquad (35)$$

has a unique solution $\eta(x)$ which is of class C' and satisfies $\eta(a)=0$.

An alternate form for (35) to which known methods of proof used in constructing standard existence theorems can be directly applied will be introduced. Let

$$\zeta_i \equiv \delta g_i = g_{iy_j}\eta_j + g_{iy_j'}\eta_j' \qquad (i=1,2,\ldots,n). \qquad (36)$$

Because by hypothesis the determinant $|g_{iy_j'}|$ is never zero, eq (36) can be solved for η_j'. The solution is

$$\eta_i' = \pi_i(x,\eta,\zeta) \qquad (i=1,2,\ldots,n) \qquad (37)$$

where π_i is a linear and homogeneous function of η_i and ζ_i. By use of (37) δh_i may be expressed as

$$\delta h_i = h_{iy_j}\eta_j + h_{iy_j'}\pi_j(x,\eta,\zeta),$$

a linear and homogeneous function in η_i and ζ_i. Therefore, any pair η, η' satisfying (35) are transformed by (36) into a pair η, ζ satisfying

$$\zeta_i - \int_a^x [h_{iy_j}\eta_j + h_{iy_j'}\pi_j(t,\eta,\zeta)]\,dt = m_i(x) \quad (i=1,2,\ldots,n)$$

$$\eta_i - \int_a^x \pi_i(t,\eta,\zeta)\,dt = 0 \qquad (i=1,2,\ldots,n). \qquad (38)$$

The last n equations of (38) are obtained by integrating (37) and recalling that $\eta_i(a)=0$.

Conversely, assume η,ζ satisfy (38) the second condition of that set implies (37). Inverting this last system of equations, one obtains (36), and by substitution in the first condition of (38) eq (35) result. Thus instead of seeking a solution of (35) which is of class C' one may equivalently seek a continuous solution of (38).

The system (38) has the form

$$\gamma_i(x) = \int_a^x A_{ij}(t)\gamma_j(t)\,dt + \overline{m}_i(x) \qquad (i=1,2,\ldots,2n) \qquad (39)$$

where A_{ij}, $\overline{m}_i$ are all continuous on $a \le x \le b$. From well known methods of proof which have been used by Bliss [2, p. 274–278] and others for systems of differential equations which can be obtained by formally differentiating (39) it can be shown that (39) has a unique continuous solution satisfying $\gamma_i(a) = \overline{m}_i(a)$. Since

$$\gamma_{n+1}(a) \equiv \eta_i(a) = 0 \qquad (i=1,2,\ldots,n)$$

the lemma is proved.

7. Existence of an Inverse of $\delta T(x,y,\eta)$

In this section the existence of a linear bounded inverse in η for the function $\delta T(x,y;\eta)$ is discussed. An idea that plays a leading role in this discussion is that of conjugate points.

DEFINITION 5. A point P_3 is said to be *conjugate* to the point P_1 on an arc joining P_1 and P_2 if there exists a solution of

$$\frac{d}{dx}\,\delta g_i(x,y;\eta) - \delta h_i(x,y;\eta) = 0 \qquad (i=1,2,\ldots,n) \qquad (40)$$

along this arc, which vanishes at P_1 and P_3 but is not identically zero between P_1 and P_3.

A consequence of definition 5 that will be apropos to our present purpose is stated in the following lemma.

LEMMA 8. *Let $\eta_i^{(j)}$ $(i,j=1,2,\ldots,n)$ be solutions of*

$$L_i(\eta)\equiv\delta g_i-\int_a^x \delta h_i\,dt=\delta_{ij},\quad \eta_i^{(j)}(a)=0 \qquad (i=1,2,\ldots,n). \tag{41}$$

Then $x=b$ is conjugate to $x=a$ if, and only if, the determinant $|\eta_i^{(j)}(b)|$ is zero.

By lemma 7 the $\eta_i^{(j)}(x)$ described in the hypothesis exist and are of class C'. In addition to satisfying (41), the $\eta_i^{(j)}$ are n linearly independent solutions of (40). Thus the n-parameter family of solutions of (40) which vanish at $x=a$ is given by

$$\eta_i(x,C_1,C_2,\ldots,C_n)\equiv\eta_i^{(j)}(x)C_j \qquad (i,j=1,2,\ldots,n). \tag{42}$$

Setting $\eta_i(b)=0$ in (42), one obtains the system of linear algebraic equations

$$\eta_i^{(j)}(b)C_j=0 \qquad (i=1,2,\ldots,n). \tag{43}$$

In order for $x=b$ to be conjugate to $x=a$, the system (43) must have a nontrivial solution. This is possible if, and only if, $|\eta_i^{(j)}(b)|=0$.

LEMMA 9. *For each admissible y in Y whose end points are not conjugate $\delta T(x,y;\eta)$ is a 1—1 mapping U onto itself.*

By lemma 5, $\delta T(x,y;\eta)$ is an element of U for each η in U and any admissible function y. It remains to be demonstrated that the system (34) has a unique solution η in U for each $l(x)$ in U provided that the end points of y are not conjugate. By lemma 7, there exists a unique $\overline{\eta}(x)$ of class C' which satisfies

$$\delta g_i-\int_a^x \delta h_i\,dt=l_i'(x),\quad \overline{\eta}_i(a)=0 \qquad (i=1,2,\ldots,n).$$

Let $\eta_i^{(j)}$ be the functions mentioned in lemma 8. Then by lemma 8 there always exists a unique set $\{C_i\}$ $(i=1,2,\ldots,n)$ such that

$$\eta_i(b)+\overline{\eta}_i^{(j)}(b)C_j=0 \qquad (i=1,2,\ldots,n).$$

Consequently, by (41),

$$\eta_i(x)\equiv\overline{\eta}_i(x)+\eta_i^{(j)}(x)C_j \qquad (i=1,2,\ldots,n) \tag{44}$$

is a solution of

$$\delta g_i-\int_a^x \delta h_i\,dt=l_i'+C_i \qquad (i=1,2,\ldots,n) \tag{45}$$

which vanishes at $x=a$ and $x=b$. That $\eta(x)$ is of class C' is apparent. That $\eta(x)$ is unique follows from the condition of nonconjugate end points. For if $\eta^{(1)}$ and $\eta^{(2)}$ both satisfy (45) and the boundary conditions, their difference must satisfy (40) and vanish at both end points of $a\leq x\leq b$, that is, their difference must be identically zero.

By integration of (45) it follows that $\eta(x)$ must satisfy

$$\int_a^b\left[\delta g_i-\int_a^t \delta h_i\,ds\right]dt-C_i(x-a)=l_i(x) \qquad (i=1,2,\ldots,n). \tag{46}$$

Letting $x=b$ in (46) one sees that

$$\int_a^x\left[\delta g_i-\int_a^t \delta h_i\,ds\right]dt-C_i(b-a)=l_i(b)=0 \qquad (i=1,2,\ldots,n).$$

Therefore,

$$C_i=\frac{1}{b-a}\int_a^b\left[\delta g_i-\int_a^t \delta h_i\,ds\right]dt \qquad (i=1,2,\ldots,n).$$

Consequently, $\eta(x)$ satisfies (34).

We conclude with a lemma that states the result we have been pointing at in this and the preceding section.

LEMMA 10. *For each admissible y in Y whose end points are not conjugate the inverse of $\delta T(x,y;\eta)$ exists and is linear and bounded at y.*

By lemma 5, and lemma 9, $\delta T(x,y;\eta)$ is a bounded linear transformation mapping a Banach space in a 1—1 manner onto itself. As is shown in Hille [3, p. 28–29], a transformation having the above properties must possess a bounded inverse.

The inverse of a linear transformation is linear.

8. Differentiability Properties of $\delta T(x,y;\eta)$

In this section two lemmas that will be useful in establishing the fact that δT is one of the class of operators that was denoted by $K(y;\eta)$ in section 4 are proved. The first lemma will help to show that δT satisfies condition (I) of section 4, whereas the second relates to conditions (II) and (IV) of the same section. First, however, we present a lemma that will considerably simplify the statement of the proofs of our two principal ones.

LEMMA 11. *Let $p(x,y,y',\eta,\xi)$ be of the form*

$$p(x,y,y',\eta,\xi)=a_{ij}(x,y,y')\eta_i\xi_j+b_{ij}(x,y,y')\eta_i\xi_i'+c_{ij}(x,y,y')\eta_i'\xi_j' \qquad (i,j=1,\ldots,n), \tag{47}$$

where a_{ij}, b_{ij}, c_{ij}, are all continuous functions on the same region R on which g and h are defined, y is an admissible element of Y, and η, ξ are elements of U. Let $y^{(0)}$ be any admissible element of Y and $\delta>0$ be such that the closure of $(y^{(0)})_\delta$ contains only admissible arcs. Then given $\epsilon>0$ it follows that there exists $d>0$ such that

$$|p(x,y^{(1)},y^{(1)\prime},\eta,\xi)-p(x,y^{(2)},y^{(2)\prime},\eta,\xi)|<\epsilon\,||\eta||\cdot||\xi|| \tag{48}$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$, which satisfy $\|y^{(1)}-y^{(2)}\|<d$ and all η,ξ in U. Furthermore, there exists a constant $C\geq 0$ such that

$$\sup_x |p(x,y,y',\eta,\xi)| \leq C\|\eta\|\cdot\|\xi\|, \tag{49}$$

and

$$\sup_x \left| \int_a^x p(x,y,y',\eta,\xi)dx \right| \leq (b-a)C\|\eta\|\cdot\|\xi\| \tag{50}$$

for y in the closure of $(y^{(0)})_\delta$, η, ξ in U.

Statements (49) and (50) follow from a simple extension of lemma 6.

In order to establish the remaining part of the lemma, let

$$\varphi(x,y^{(1)},y^{(2)},\eta,\xi) = p(x,y^{(1)},y^{(1)'},\eta,\xi) - p(x,y^{(2)},y^{(2)'},\eta,\xi).$$

Then φ is a continuous function on the closed and bounded set $S: a\leq x\leq b$, $y^{(1)}$, $y^{(2)}$ in the closure of $(y^{(0)})_\delta$, $\|\eta\|=\|\xi\|=1$. That it is sufficient to merely consider η, ξ on the unit sphere follows from the bilinearity of p. It is clear that $\varphi=0$ on the subset of S for which $y^{(1)}=y^{(2)}$. Thus $|\varphi|<\epsilon$ on a neighborhood of this set, that is, on $a\leq x\leq b$, $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$, $\|y^{(1)}-y^{(2)}\|<d$, $\|\eta\|=\|\xi\|=1$ for d sufficiently small. This proves lemma 11.

LEMMA 12. *For each admissible $y^{(0)}$ in Y there exists a constant $\delta>0$ such that $\delta T(x,y;\eta)$ is of class C' in y uniformly on $(y^{(0)})_\delta$.*

Let $\delta^2 g_i(x,y;\eta,\xi)$ and $\delta^2 h_i(x,y;\eta,\xi)$ be defined by

$$\delta^2 g_i \equiv g_{v_jv_k}(x,y,y')\eta_j\xi_k + g_{v_jv_k'}(x,y,y')\eta_j\xi_k' + g_{v_j'v_k}\eta_j'\xi_k + g_{v_j'v_k'}\eta_j'\xi_k', \tag{51}$$

and

$$\delta^2 h_i \equiv h_{v_jv_k}(x,y,y')\eta_j\xi_k + h_{v_jv_k'}\eta_j\xi_k' + h_{v_j'v_k}\eta_j'\xi_k + h_{v_j'v_k'}\eta_j'\xi_k' \tag{52}$$

where y is some admissible element of Y and η,ξ are arcs in U. Then the function $d(\delta T)$ mentioned in definition 4 can be defined as

$$d(\delta T_i) \equiv \delta^2 T_i(x,y;\eta,\xi) = \int_a^t \left[\delta^2 g_i - \int_a^t \delta^2 h_i ds\right] dt - \frac{x-a}{b-a}\int_a^b \left[\delta^2 g_i - \int_a^t \delta^2 h_i ds\right] dt \qquad (i=1,2,\ldots,n).$$

By reasoning similar to that used in considering δT it follows that $\delta^2 T$ is on Y_0UU to U. That $\delta^2 T$ is bilinear in η and ξ is obvious. To complete the proof of the lemma conditions (1), (2), and (3) of definition 4 must be established.

(1) *For every admissible $y^{(0)}$ in Y and every finite $\delta>0$ such that the closure of $(y^{(0)})_\delta$ contains only admissible arcs $\delta^2 T(x,y;\eta,\xi)$ is continuous in y uniformly on $(y^{(0)})_\delta$.*

Let $F(x,y,y';\eta,\xi)$ be any function defined on Y and U. Then we define

$$\Delta F(1,2) \equiv F(x,y^{(1)},y^{(1)'};\eta,\xi) - F(x,y^{(2)},y^{(2)'};\eta,\xi).$$

Keeping this convention in mind, one may write

$$\begin{aligned}
&\sup_x |\delta^2 T_i(x,y^{(1)};\eta,\xi) - \delta^2 T_i(x,y^{(2)};\eta,\xi)| \\
&= \sup_x \left| \int_a^x \left[\Delta\delta^2 g_i(1,2) - \int_a^t \Delta\delta^2 h_i(1,2)\,ds\right] dt \right. \\
&\qquad \left. - \frac{x-a}{b-a}\int_a^b \left[\Delta\delta^2 g_i(1,2) - \int_a^t \Delta\delta^2 h_i(1,2)\,ds\right] dt \right| \qquad (53) \\
&\leq 2\int_a^b \left|\Delta\delta^2 g_i(1,2) - \int_a^t \Delta\delta^2 h_i(1,2)\,ds\right| dt \\
&\leq 2\int_a^b |\Delta\delta^2 g_i(1,2)|\,dt + 2(b-a)\int_a^b |\Delta\delta^2 h_i(1,2)|\,dt.
\end{aligned}$$

Examination of (51) and (52) reveals that both $\delta^2 g_i$ and $\delta^2 h_i$ are of the type $p(x,y,y',\eta,\xi)$ defined in lemma 11. Thus application of lemma 11 to (53) yields the conclusion

$$\sup_x |\delta^2 T_i(x,y^{(1)};\eta,\xi) - \delta^2 T_i(x,y^{(2)};\eta,\xi)| < \epsilon\|\eta\|\cdot\|\xi\|$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ which satisfy a condition of the form $\|y^{(1)}-y^{(2)}\|\leq d$. The same result can be established when $\delta^2 T_i$ is replaced by $\delta^2 T_i'$. Therefore, (1) follows at once from lemma 4.

(2) *For every admissible $y^{(0)}$ in Y there exists a $\delta>0$ such that $\delta^2 T(x,y;\eta,\xi)$ is uniformly bounded on $(y^{(0)})_\delta$.*

From an application of (49) and (50) of lemma 11 the inequality

$$\begin{aligned}
\sup_x |\delta^2 T_i(x,y^{(0)};\eta,\xi)| &\leq 2\int_a^b \left|\delta g_i - \int_a^t \delta h_i ds\right| dt \\
&\leq 2\int_a^b \left[\sup_t |\delta g_i| + \sup_t \left|\int_a^t \delta h_i ds\right|\right] dt \\
&\leq C^{(i)}\|\eta\|\cdot\|\xi\|, \quad C^{(i)}\geq 0 \qquad (54)
\end{aligned}$$

is obtained. From analogous considerations the existence of $C^{(i)'}\geq 0$ such that

$$\sup_x |\delta^2 T_i'(x,y^{(0)};\eta,\xi)| \leq C^{(i)'}\|\eta\|\cdot\|\xi\| \tag{55}$$

can be demonstrated. Because of lemma 4, inequalities (54) and (55) imply that $\delta^2 T$ is bounded in y at $y^{(0)}$. By (1) $\delta^2 T$ is continuous in y at $y^{(0)}$. Thus by lemma 1 there exists $\delta>0$ such that $\delta^2 T$ is uniformly bounded in y on $(y^{(0)})_\delta$.

(3) *For each admissible $y^{(0)}$ in Y and each finite $\delta>0$ which is such that the closure of $(y^{(0)})_\delta$ contains only admissible arcs, the function $R(y^{(1)},y^{(2)};\eta)$ defined by replacing F by δT and dF by $\delta^2 T$ in (4) is*

such that given $\epsilon>0$ there exists $d>0$ for which

$$\|R(y^{(1)},y^{(2)};\eta)\|<\epsilon\|\eta\|$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ satisfying $\|y^{(2)}-y^{(1)}\|\leq d$ and all η in U.

Consider

$$\delta T_i(x,y^{(1)};\eta)-\delta T_i(x,y^{(2)};\eta)=\int_a^x[\Delta g_{ty_j}(1,2)\eta_j$$
$$+\Delta g_{ty_j'}(1,2)\eta_i'-\int_a^t(\Delta h_{ty_j}(1,2)\eta_j+\Delta h_{ty_j'}(1,2)\eta_i')ds]dt$$
$$-\frac{x-a}{b-a}\int_a^b[\Delta g_{ty_j}(1,2)\eta_j+\Delta g_{ty_j'}(1,2)\eta_i'$$
$$-\int_a^t(\Delta h_{ty_j}(1,2)\eta_j+\Delta h_{ty_j'}(1,2)\eta_i')ds]dt. \quad (56)$$

Applying the law of the mean to a typical Δ-symbol in (56), one obtains

$$\Delta g_{ty_j}(1,2)=g_{ty_jy_k}(x,y^{(3)},y^{(3)\prime})(y_k^{(1)}-y_k^{(2)})$$
$$+g_{ty_jy_k'}(x,y^{(3)},y^{(3)\prime})(y_k^{(1)\prime}-y_k^{(2)\prime}),$$

where

$$y^{(3)}=y^{(2)}+\mu(y^{(1)}-y^{(2)}),\qquad 0<\mu<1.$$

Thus by applying the law of the mean to the remaining Δ-symbols and taking note of (51) and (52), we may write

$$\delta T_i(x,y^{(1)};\eta)-\delta T_i(x,y^{(2)};\eta)-\delta^2T_i(x,y^{(1)};\eta,y^{(1)}-y^{(2)})$$
$$=\int_a^x\boldsymbol{D}(x,y;\eta,\xi)dt-\frac{x-a}{b-a}\int_a^b\boldsymbol{D}dt,$$

where

$$\boldsymbol{D}\equiv\Delta g_{ty_jy_k}(3,1)\eta_j(y_k^{(1)}-y_k^{(2)})+\Delta g_{ty_jy_k'}(3,1)\eta_j(y_k^{(1)\prime}-y_k^{(2)\prime})$$
$$+\Delta g_{ty_j'y_k}(3,1)\eta_i'(y_k^{(1)}-y_k^{(2)})+\Delta g_{ty_j'y_k'}(3,1)\eta_i'(y_k^{(1)\prime}-y_k^{(2)\prime})$$
$$-\int_a^t[\Delta h_{ty_jy_k}(3,1)\eta_j(y_k^{(1)}-y_k^{(2)})+\Delta h_{ty_jy_k'}(3,1)\eta_j\times$$
$$(y_k^{(1)\prime}-y_k^{(2)\prime})$$
$$+\Delta h_{ty_j'y_k}(3,1)\eta_i'(y_k^{(1)}-y_k^{(2)})+\Delta h_{ty_j'y_k'}(3,1)\eta_i'\times$$
$$(y_k^{(1)\prime}-y_k^{(2)\prime})]ds. \quad (57)$$

Of course, the 3 appearing in the arguments of the various Δ-symbols does not always refer to the same value of μ.

The function $g_{ty_jy_k}\eta_j(y_k^{(1)}-y_k^{(2)})$, that is typical of the functions to which the Δ operation is applied in (57), is of the type considered in lemma 11. Thus it readily follows from that lemma that,

$$\sup_x|\delta T_i(x,y^{(1)};\eta)-\delta T_i(x,y^{(2)};\eta)-\delta^2T_i(x,y^{(1)};\eta,y^{(1)}-$$
$$y^{(2)})|\leq 2\int_a^b\left|\boldsymbol{D}-\int_a^t\boldsymbol{D}ds\right|dt<\epsilon\|\eta\|\cdot\|y^{(1)}-y^{(2)}\|$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ such that $\|y^{(1)}-y^{(2)}\|\leq d$. A similar result holds if δT_i and δ^2T_i are replaced by their derivatives. Therefore, by lemma 4 it is true that

$$\|R(y^{(1)},y^{(2)};\eta)\|\cdot\|y^{(1)}-y^{(2)}\|$$
$$=\|\delta T(x,y^{(1)};\eta)-\delta T(x,y^{(2)};\eta)-\delta^2T(x,y^{(2)};\eta,y^{(1)}-y^{(2)})\|$$
$$<\epsilon\|\eta\|\cdot\|y^{(1)}-y^{(2)}\| \quad (58)$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\delta$ that are close enough together. Statement (3) follows at once from (58).

By choosing the δ's of (1) and (3) to be equal to each other and less than the δ mentioned in (2), one obtains a δ that is suitable to be used in lemma 12.

LEMMA 13. *For any admissable $y^{(0)}$ in Y there exist constants $\alpha>0$, $L'\geq 0$ such that*

$$\|\delta T(x,y^{(1)};y^{(1)}-y^{(2)})-T(x;y^{(1)})+T(x;y^{(2)})\|$$
$$\leq L'\|y^{(1)}-y^{(2)}\|^2$$

for all $y^{(1)}$, $y^{(2)}$ in $y^{(0)}{}_\alpha$.

Consider

$$T_i(x;y^{(2)})-T_i(x;y^{(1)})$$
$$=\int_a^t\left[\Delta g_i(2,1)-\int_a^t\Delta h_i(2,1)ds\right]dt-$$
$$\frac{x-a}{b-a}\int_a^t\left[\Delta g_i(2,1)-\int_a^t\Delta h_i(2,1)ds\right]dt.$$

By the law of the mean

$$\Delta g_i(2,1)=g_{ty_j}(x,y^{(3)},y^{(3)\prime})(y_j^{(2)}-y_j^{(1)})+$$
$$g_{ty_j'}(x,y^{(3)},y^{(3)\prime})(y_j^{(2)\prime}-y_j^{(1)\prime})$$

where

$$y^{(3)}=y^{(1)}+\mu(y^{(2)}-y^{(1)}),\qquad 0<\mu<1.$$

As a similar result holds for Δh_i, we may write

$$\delta T_i(x,y^{(1)};y^{(1)}-y^{(2)})-T_i(x;y^{(1)})+T_i(x;y^{(2)})-$$
$$\int_a^x Z(x,y;\eta,\xi)dx=\frac{x-a}{b-a}\int_a^b Zdx,$$

where

$$Z\equiv\Delta g_{ty_j}(1,3)(y_j^{(1)}-y_j^{(2)})+\Delta g_{ty_j'}(1,3)(y_j^{(1)\prime}-y_j^{(2)\prime})-$$
$$\int_a^t[\Delta h_{ty_j}(1,3)(y_j^{(1)}-y_j^{(2)})+\Delta h_{ty_j'}(1,3)(y_j^{(1)}-y_j^{(2)\prime})]ds.$$

Of course, the 3 appearing in the arguments of Δg_i and Δh_i does not always refer to the same value of μ. Application of the law of the mean to a typical term of Z yields

$$\Delta g_{ty_j'}(1,3)=g_{ty_j'y_k}(x,y^{(4)},y^{(4)\prime})(y_j^{(1)\prime}-y_j^{(2)\prime})(y_k^{(1)}-y_k^{(2)})\mu$$
$$+g_{ty_j'y_k'}(x,y^{(4)},y^{(4)\prime})(y_j^{(1)\prime}-y_j^{(2)\prime})(y_k^{(1)\prime}-y_k^{(2)\prime})\mu,$$

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where

$$y^{(4)}=y^{(1)}+\mu^*(y^{(1)}-y^{(2)}), \qquad 0<\mu^*<1.$$

It will be noted that $\Delta g_{iy_j'}(1,3)$ is of the same form as the functions considered in lemma 11. Let α be chosen small enough so that for any $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_\alpha$, $y^{(4)}$ lies in some fixed neighborhood $(y^{(0)})_\alpha$ whose closure contains only admissible functions. Then because of our previous hypothesis on the class of h_i and g_i, we may apply lemma 11 from which the existence of $C^{(i)}\geq 0$ such that

$$\sup_x |\delta T_i(x,y^{(1)};y^{(1)}-y^{(2)})-T_i(x,y^{(1)})+T_i(x;y^{(2)})| \leq C^{(i)}||y^{(1)}-y^{(2)}||^2$$

follows. As similar considerations are valid when $\delta T_i, T_i$ are replaced by $\delta T_i', T_i'$, respectively, the conclusion of lemma 13 follows from lemma 4.

9. Convergence of the Sequence of Estimates Derived by Newton's Method

The main theorem on the convergence of the sequence of estimates derived from Newton's method is the following.

THEOREM 4. *Let $y^{(0)}$ be an admissible element of Y having nonconjugate end points. Then if there exists a sufficiently small constant $D>0$ such that $\|T(x,y^{(0)})\|<D$, the sequence of estimates derived from Newton's method will converge quadratically to a unique arc $\overline{y}$ of class C^3 which lies within a predetermined neighborhood of $y^{(0)}$ and satisfies $T(x;\overline{y})=\theta, \overline{y}(a)=y^{(0)}(a), \overline{y}(b)=y^{(0)}(b)$.*

By lemma 12, there exists a constant $\alpha_1>0$ such that $\delta T(x,y;\eta)$ is of class C' in y uniformly on $(y^{(0)})_{\alpha_1}$. By lemma 10, $\delta T^{-1}(x,y^{(0)};\eta)$ exists, and is linear and bounded at $y^{(0)}$. By lemma 5, δT is on $(y^{(0)})_{\alpha_1}U$ to U and is bounded in y and linear in η. Therefore, by lemma 3, there exists α_2 such that δT^{-1} exists on $(y^{(0)})_{\alpha_2}$ and is of class C' in y uniformly on $(y^{(0)})_{\alpha_2}$. Thus by lemma 2, there exist nonnegative constants α_3 and M such that for $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_{\alpha_2}$ and $||y^{(1)}-y^{(2)}||\leq\alpha_3$ it follows that

$$||\delta T^{-1}(x,y^{(1)};\eta)-\delta T^{-1}(x,y^{(2)};\eta)||\leq M||y^{(1)}-y^{(2)}||\cdot||\eta||.$$

Consequently, if we choose $\alpha_4\equiv\alpha_3/2$, a part of condition (I) of section 4 is satisfied on $(y^{(0)})_{\alpha_4}$.

By an argument similar to that applied in section 4, it can be shown that there exist nonnegative constants α_5, B such that

$$||\delta T^{-1}(x,y;\eta)||\leq B||\eta|| \tag{59}$$

for all y in $(y^{(0)})_{\alpha_5}$ and all η in U. If α_6 is taken to be less than or equal inf (α_4,α_5), and such that $(y^{(0)})_{\alpha_6}$ contains only admissible arcs, the part of condition (I) previously proved and inequality (59) are still valid on $(y^{(0)})_{\alpha_6}$, that is, condition (I) holds on $(y^{(0)})_{\alpha_6}$.

The existence of nonnegative constants H, L, N, which along with the nonnegative constants B, M already determined in this section satisfy relation (21) is clear. By lemma 13 there also exist α_7 and L' such that

$$||\delta T(x,y^{(1)};y^{(1)}-y^{(2)})-T(x;y^{(1)})+T(x;y^{(2)})||\leq L'||y^{(1)}-y^{(2)}||^2$$

for all $y^{(1)}$, $y^{(2)}$ in $(y^{(0)})_{\alpha_7}$. That is, condition (IV) of section 4 is valid on $(y^{(0)})_{\alpha_7}$. Thus we may choose $\alpha_8\leq\alpha_7$ such that conditions (I) and (IV) both hold on $(y^{(0)})_{\alpha_8}$. By continuity there exists α_9 such that $\|T(x;y)\|\leq D$ on $(y^{(0)})_{\alpha_9}$. Let C be such that $0<C<\inf(\alpha_8,\alpha_9)$. Then if the constant D is less than or equal to the smaller of N and $(1-H)C/B$, condition (III) is satisfied on $(y^{(0)})_{\alpha_9}$. Consequently, if the positive constant γ in theorem 3 is defined to be the inf (α_8,α_9), we may conclude by that theorem that Newton's method converges quadratically to a unique element $\overline{y}$ of $(y^{(0)})_\alpha$, $\alpha\leq\gamma$, which is such that (3) is satisfied. That

$$\overline{y}(a)=y^{(0)}(a), \qquad \overline{y}(b)=y^{(0)}(b)$$

follows at once from the fact that all the variations $\eta^{(i)}$ vanish at $x=a$ and $x=b$.

The arc $\overline{y}$ is of class C' by virtue of being in Y. To see that $\overline{y}$ is actually of class C^3 consider the equations in (x,v)

$$G_i(x,v)\equiv g_i(x,\overline{y}(x),v)-\int_a^x h_i(t,\overline{y}(t),\overline{y}'(t))dt- \frac{1}{b-a}\int_a^b\Big[g_i(x,\overline{y}(x),\overline{y}'(x))- \int_a^t h_i(s,\overline{y}(s),\overline{y}'(s))ds\Big]dt=0 \quad (i=1,2,\ldots,n). \tag{60}$$

By what just been proved the point $(x,\overline{y}'(x))$ is a particular solution of (60) for each x on $a\leq x\leq b$. Since for each such x, $(x,\overline{y}(x),\overline{y}'(x))$ is an interior point of the region R in which g and h are of class C^2, $G(x,v)$ is of the same class in a neighborhood of each point $(x,\overline{y}'(x))$. By hypothesis the Jacobian

$$|G_{iv_j}|=|g_{iy_j'}|\neq 0$$

in the region R. Thus the implicit function theorem (see Bliss [2, p. 269]) is applicable. This theorem tells us that $\overline{y}'(x)$ is of the same class as $G(x,v)$, that is, $\overline{y}(x)$ is of class C^3.

The following theorem gives further conditions under which the conclusions of theorem 4 will be valid.

THEOREM 5. *Let y^* be an admissible arc with nonconjugate end points such that $T(x;y^*)=\theta$. Then there exists a constant $\alpha>0$ such that for any initial estimate $y^{(0)}$ taken from $(y^*)_\alpha$ the conclusions of theorem 4 are valid.*

By reasoning similar to that employed in the proof of theorem 4 there exists $\beta>0$ such that conditions (I) and (II) and (IV) are satisfied on $(y^*)_\beta$. By continuity β may be made to have the additional property that $\|T(y)\|\leq N$ for all y in $(y^*)_\beta$. Let α be such that $0<\alpha<\beta/2$. Then for any $y^{(0)}$ belonging to $(y^*)_\alpha$ it follows that $(y^{(0)})_{\beta/2}$ is contained in $(y^*)_\beta$. Thus (I), (II), (IV), and the inequality $\|T(y)\|\leq N$ hold on $(y^{(0)})_{\beta/2}$. Let C be such that $0<C<\beta/2$. Then by continuity α may be taken so small that for all $y^{(0)}$ in $(y^*)_\alpha$ $\|T(y^{(0)})\|\leq(1-H)C/B$. This says condition (III) also holds on the $(\beta/2)$-neighborhood of any $y^{(0)}$ taken from $(y^*)_\alpha$. We now invoke theorem 3 and the remainder of the proof follows as in theorem 4.

If one wishes, he may hold $y^{(0)}$ fixed in $\delta T(x,y;\eta)$ throughout the entire process. Theorem 2 is applicable in this simplified case, and with its help all of the previous results can be shown to be true for the simplified Newton's method.

10. A Lemma Concerning Conjugate Points

It is easily seen that the equation

$$\delta T(x,y;\eta)=\int_a^x\left[\delta g_i(t,y;\eta)-\int_a^t\delta h_i(s,y;\eta)ds\right]dt-\frac{x-a}{b-a}\int_a^b\left[\delta g_i-\int_a^t\delta h_i ds\right]dt=0 \qquad (i=1,2,\ldots,n) \tag{61}$$

and eq (40) of definition 5 are equivalent. Thus in the considerations of this section it will be assumed that (40) has been replaced by (61) in the definition of conjugate points.

If y is an arc such that $\delta T^{-1}(x,y;\eta)$, the inverse with respect to η of $\delta T(x,y;\eta)$, exists for all η in U, then y has nonconjugate end points, because the unique element of U which $\delta T(x,y;\eta)$ maps into θ is θ itself. In lemma 9 it was shown that if y has nonconjugate end points, $\delta T^{-1}(x,y;\eta)$ exists. Thus a necessary and sufficient condition for $\delta T^{-1}(x,y;\eta)$ to exist for all η in U is that y have nonconjugate end points. However, by lemma 3, if δT^{-1} exists at $y^{(0)}$ it must exist in some neighborhood of $y^{(0)}$. Therefore, the following lemma may be looked upon as a corollary of lemma 3.

LEMMA 14. *Let $y^{(0)}$ be an admissible element of Y having nonconjugate end points. Then there exists $\delta>0$ such that no elements of $(y^{(0)})_\delta$ have conjugate end points.*

In this section we wish to give an alternate proof of lemma 14, which depends directly on the properties of functions with Lebesgue square integrable derivatives rather than on the Banach space theory of section 3.

To construct a proof of lemma 14 by contradiction, assume that there exists no neighborhood of $y^{(0)}$ in which all elements have nonconjugate end points. Then in every neighborhood of $y^{(0)}$ there exists at least one function y such that (61) has a nontrivial solution along y which vanishes at $x=a$ and $x=b$. Consequently, from consideration of the $1/q$-neighborhoods of $y^{(0)}$ for $q=1,2,\ldots$ one obtains the sequences $\{y^{(q)}\}$, $\{\eta^{(q)}\}$, which are such that

$$\left.\begin{aligned} &y^{(q)}(x)\to y^{(0)}(x) \quad \text{uniformly on } a\leq x\leq b, \quad (q=1,2,\ldots)\\ &y^{(q)\prime}(x)\to y^{(0)\prime}(x) \quad \text{uniformly on } a\leq x\leq b, \quad (q=1,2,\ldots)\end{aligned}\right\} \tag{62}$$

$$\left.\begin{aligned} &\delta T(x,y^{(q)};\eta^{(q)})=\theta \qquad (q=1,2,\ldots)\\ &\eta^{(q)}(a)=\eta^{(q)}(b)=\theta \qquad (q=1,2,\ldots).\end{aligned}\right\} \tag{63}$$

Since all arcs η under consideration belong to the space U they must certainly belong to the class of the totality of arcs in $(x,\eta_1,\ldots,\eta_n)$-space defined by a set of n real valued functions

$$\eta_i(x) \qquad (a\leq x\leq b;\quad i=1,2,\ldots,n)$$

that are absolutely continuous and have integrable square derivatives $\eta'(x)$ on $a\leq x\leq b$. Henceforth, we will think of the η's as belonging to this class.

Without loss of generality one can choose $\eta^{(q)}$ such that

$$\int_a^b\eta_i^{(q)\prime}\cdot\eta_i^{(q)\prime}dx=1 \qquad (q=1,2,\ldots) \qquad \text{(no sum on } q). \tag{64}$$

Equations (64) imply the existence of a subsequence $\{\eta^{(q_j)\prime}\}$ which converges weakly in L^2 to some element $\eta^{(0)\prime}$. We can assume that this subsequence is the same as our original sequence. The weak convergence in L^2 of the sequence $\{\eta^{(q)\prime}\}$ implies

$$\eta_i^{(q)}\to\eta_i^{(0)} \quad \text{uniformly on } a\leq x\leq b \qquad (i=1,2,\ldots,n) \tag{65}$$

$$\eta_i^{(q)\prime}\to\eta_i^{(0)\prime} \quad \text{weakly in } L^2 \qquad (i=1,2,\ldots,n).$$

It is also true that

$$\int_a^b\eta_i^{(0)\prime}\cdot\eta_i^{(0)\prime}dx\leq\liminf_{q\to\infty}\int_a^b\eta_i^{(q)\prime}\eta_i^{(q)\prime}dx \qquad \text{(no sum on } q). \tag{66}$$

A proof of this last fact has been given by McShane [4, p. 355].

Inequality (66) will be used in the proof of the following lemma.

LEMMA 15. *Let $\{\eta^{(q)\prime}\}$ be a sequence in L^2 such that (64) and (65) are satisfied and let $A_{ij}^{(q)}(x)$, $B_{ij}^{(q)}(x)$ $(i,j=1,2,\ldots,n)$ be continuous functions for almost all q such that*

$$A_{ij}^{(q)}\to A_{ij}^{(0)} \quad \text{uniformly on } a\leq x\leq b \qquad (i,j=1,2,\ldots,n) \tag{67}$$

$B_{ij}^{(q)} \to B_{ij}^{(0)}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$

Then if

$$W_i^{(q)}(x)=\int_a^x (A_{ij}^{(q)}\eta_j^{(q)}+B_{ij}^{(q)}\eta_j^{(q)\prime})dt \quad (q=1,2,\ldots,\ i=1,2,\ldots,n) \quad \text{(no sum on } q),$$

it follows that

$$W_i^{(q)} \to W_i^{(0)} \quad \text{uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n) \tag{68}$$

$$W_i^{(q)\prime} \to W_i^{(0)\prime} \quad \text{weakly in } L^2 \quad (i=1,2,\ldots,n)$$

Consider

$$|W_i^{(q)}(x)-W_i^{(0)}(x)|=$$

$$\left|\int_a^x [(A_{ij}^{(q)}\eta_j^{(q)}-A_{ij}^{(0)}\eta_j^{(0)})+(B_{ij}^{(q)}\eta_j^{(q)\prime}-B_{ij}^{(0)}\eta_j^{(0)\prime})]dt\right| \leq$$

$$\left|\int_a^x (A_{ij}^{(q)}\eta_j^{(q)}-A_{ij}^{(0)}\eta_j^{(q)})dt\right|+\left|\int_a^x (A_{ij}^{(0)}\eta_j^{(q)}-A_{ij}^{(0)}\eta_j^{(0)})dt\right|+$$

$$\left|\int_a^x (B_{ij}^{(q)}\eta_j^{(q)\prime}-B_{ij}^{(0)}\eta_j^{(q)\prime})dt\right|+\left|\int_a^x (B_{ij}^{(0)}\eta_j^{(q)\prime}-B_{ij}^{(0)}\eta_j^{(0)\prime})dt\right|.$$

The first of the four terms on the right of the sign of inequality satisfies

$$\left|\int_a^x (A_{ij}^{(q)}\eta_j^{(q)}-A_{ij}^{(0)}\eta_j^{(q)})dt\right| \leq \int_a^b |A_{ij}^{(q)}-A_{ij}^{(0)}|\cdot|\eta_j^{(q)}|dt$$

$$\leq (b-a)\max|A_{ij}^{(q)}-A_{ij}^{(0)}|\cdot|\eta_i^{(q)}|.$$

By (67) $\max|A_{ij}^{(q)}-A_{ij}^{(0)}|\to 0$ for each $(i,j=1,2,\ldots,n)$. By (65) $|\eta_j^{(q)}|$ is uniformly bounded for all j, q under consideration. Thus the first term goes uniformly to zero on $a \leq x \leq b$. The second term on the right satisfies

$$\left|\int_a^x (A_{ij}^{(0)}\eta_j^{(q)}-A_{ij}^{(0)}\eta_j^{(0)})dt\right| \leq \int_a^b |A_{ij}^{(0)}|\cdot|\eta_j^{(q)}-\eta_j^{(0)}|dt$$

$$\leq \max|A_{ij}^{(0)}|\int_a^b |\eta_j^{(q)}-\eta_j^{(0)}|dt.$$

This goes uniformly to zero by (65). For the third term we have

$$\left|\int_a^x (B_{ij}^{(q)}\eta_j^{(q)\prime}-B_{ij}^{(0)}\eta_j^{(q)\prime})dt\right| \leq \int_a^b |B_{ij}^{(q)}-B_{ij}^{(0)}|\cdot|\eta_j^{(q)\prime}|dt$$

$$\leq \max|B_{ij}^{(q)}-B_{ij}^{(0)}|\int_a^b |\eta_i^{(q)\prime}|dt.$$

By the Schwartz inequality and (64)

$$\left(\int_a^b |\eta_i^{(q)\prime}|dt\right)^2 \leq (b-a)\int_a^b \eta_i^{(q)\prime 2}dt \leq (b-a) \quad (q=1,2,\ldots).$$

Thus $\int_a^b |\eta_j^{(q)\prime}|dt$ is uniformly bounded for $(j=1,2,\ldots)$ $(q=1,2,\ldots)$. Consequently, one may conclude from (67) that the third term goes uniformly to zero on $a \leq x \leq b$.

Only the fourth term remains to be discussed. Let j be temporarily held fixed. As is well known there exists for each arbitrary $\epsilon>0$ a polynomial $P(x)$ such that

$$\int_a^b (B_{ij}^{(0)}(x)-P(x))^2 dx \leq \frac{\epsilon^2}{16}. \tag{69}$$

By (64), (66), and the Minkowski inequality it is true that

$$\left(\int_a^b [\eta_i^{(q)\prime}-\eta_i^{(0)\prime}]^2 dx\right)^{\frac{1}{2}} \leq$$

$$\left(\int_a^b (\eta_i^{(q)\prime})^2 dx\right)^{\frac{1}{2}}+\left(\int_a^b (\eta_i^{(0)\prime})^2 dx\right)^{\frac{1}{2}} \leq 2. \tag{70}$$

Thus by use of Schwarz' inequality, (69), and (70) one obtains

$$\left|\int_a^x B_{ij}^{(0)}(\eta_i^{(q)}-\eta_i^{(0)})'dt-\int_a^x P(t)(\eta_i^{(q)}-\eta_i^{(0)})'dt\right|$$

$$\leq \left(\int_a^x (B_{ij}^{(0)}-P)^2 dt\right)^{\frac{1}{2}}\left(\int_a^x (\eta_i^{(q)\prime}-\eta_i^{(0)\prime})^2 dt\right)^{\frac{1}{2}} \tag{71}$$

$$\leq \left(\int_a^b (B_{ij}^{(0)}-P)^2 dt\right)^{\frac{1}{2}}\left(\int_a^b (\eta_i^{(q)\prime}-\eta_i^{(0)\prime})\,dt\right)^{\frac{1}{2}} \leq \frac{\epsilon}{2}$$

for all x on $a \leq x \leq b$. Integration by parts yields

$$\int_a^x P(t)(\eta_i^{(q)}-\eta_i^{(0)})'dt$$

$$=P(x)(\eta_i^{(q)}-\eta_i^{(0)})-\int_a^x P'(t)(\eta_i^{(q)}-\eta_i^{(0)})\,dt.$$

Therefore, it follows from the uniform convergence of $(\eta_i^{(q)}-\eta_i^{(0)})$ to zero that

$$\left|\int_a^x P(t)(\eta_i^{(q)\prime}-\eta^{(0)\prime})dt\right| \leq \frac{\epsilon}{2} \quad \text{uniformly on } a \leq x \leq b \tag{72}$$

for q sufficiently large. Formulas (71) and (72) imply that

$$\left|\int_a^x B_{ij}^{(0)}(\eta_i^{(q)\prime}-\eta_i^{(0)\prime})dt\right| \to 0 \quad \text{uniformly on } a \leq x \leq b.$$

This completes the proof of the first statement of (68).

Let l be any element of L^2 and consider

$$\left|\int_a^b W^{(\varphi)\prime} l\,dx - \int_a^b W^{(0)\prime} l\,dx\right| \leq$$

$$\left|\int_a^b (A_{ij}^{(\varphi)}\eta_j^{(\varphi)} - A_{ij}^{(0)}\eta_j^{(0)})l\,dx\right| + \left|\int_a^b (B_{ij}^{(\varphi)}\eta_j^{(\varphi)\prime} - B_{ij}^{(0)}\eta_j^{(0)\prime})l\,dx\right|$$

$$\leq \max|l|\left\{\int_a^b |A_{ij}^{(\varphi)}\eta_j^{(\varphi)} - A_{ij}^{(0)}\eta_j^{(0)}|\,dx + \int_a^b |B_{ij}^{(\varphi)}\eta_j^{(\varphi)\prime} - B_{ij}^{(0)}\eta_j^{(\varphi)\prime}|\,dx\right| +$$

$$\left|\int_a^b l B_{ij}^{(0)}(\eta_j^{(\varphi)\prime} - \eta_j^{(0)\prime})\,dx\right|.$$

By a repetition of the arguments already given above the terms multiplied by $\max|l|$ go to zero. The factor l may be lumped with $B_{ij}^{(0)}$ without changing the reasoning used in analyzing term four. Thus the second statement of (68) is seen to be true.

The following corollary is a direct result of lemma 15.

COROLLARY 1. *Let the hypothesis of lemma 15 hold and let*

$$U_i^{(\varphi)}(x) \equiv W_i^{(\varphi)}(x) - \frac{x-a}{b-a} W_i^{(\varphi)}(b) \tag{73}$$

$$V_i^{(\varphi)}(x) \equiv \int_a^x W_i^{(\varphi)}(t)dt - \frac{x-a}{b-a}\int_a^b W_i^{(\varphi)}(t)dt. \tag{74}$$

Then,

$U_i^{(\varphi)} \to U_i^{(0)}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$

$V_i^{(\varphi)} \to V_i^{(0)}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$

$U_i^{(\varphi)\prime} \to U_i^{(0)\prime}$ weakly in L^2 $(i=1,2,\ldots,n)$

$V_i^{(\varphi)\prime} \to V_i^{(0)\prime}$ uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$.

Let

$$u_i(x,y;\eta) \equiv \int_a^x \delta g_i\,dt - \frac{x-a}{b-a}\int_a^b \delta g_i\,dt \quad (i=1,2,\ldots,n)$$

$$v_i(x,y;\eta) \equiv \int_a^x \left(\int_a^t \delta h_i\,ds\right)dt - \frac{x-a}{b-a}\int_a^b \left(\int_a^t \delta h_i\,ds\right)dt \quad (i=1,2,\ldots,n).$$

Then u and v are elements of U satisfying

$$\delta T_i(x,y;\eta) = u_i - v_i \qquad (i=1,2,\ldots,n). \tag{75}$$

Since $\int_a^x \delta g_i(t,y^{(\varphi)};\eta^{(\varphi)})dt$ and $\int_a^x \delta h_i(t,y^{(\varphi)};\eta^{(\varphi)})dt$ are functions of the type $W_i^{(\varphi)}(x)$ defined in lemma 15 for each admissible $y^{(\varphi)}$ in Y and $\eta^{(\varphi)}$ in L^2, $u_i(x,y^{(\varphi)};\eta^{(\varphi)})$ and $v_i(x,y^{(\varphi)};\eta^{(\varphi)})$ are respectively of the types (73) and (74). Consequently, if we take $\{y^{(\varphi)}\}$ and $\{\eta^{(\varphi)}\}$ to be the sequences which result from contradicting lemma 14, the hypotheses of lemma 15 will be fulfilled because of (62) and (65), and corollary 1 will then imply

$$u_i(x,y^{(\varphi)};\eta^{(\varphi)}) \to u_i(x,y^{(0)};\eta^{(0)})$$
uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$ (76)

$$v_i(x,y^{(\varphi)};\eta^{(\varphi)}) \to v_i(x,y^{(0)};\eta^{(0)})$$
uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$ (77)

$$u_i'(x,y^{(\varphi)};\eta^{(\varphi)}) \to u_i'(x,y^{(0)};\eta^{(0)})$$
weakly in L^2 $(i=1,2,\ldots,n)$ (78)

$$v_i'(x,y^{(\varphi)};\eta^{(\varphi)}) \to v_i'(x,y^{(0)};\eta^{(0)})$$
uniformly on $a \leq x \leq b$ $(i=1,2,\ldots,n)$. (79)

In view of (63), formulas (76) and (77) yield the result

$$\delta T(x,y^{(0)};\eta^{(0)}) = \theta. \tag{80}$$

The limit function $\eta^{(0)}(x)$ is clearly a continuous function on $a \leq x \leq b$ satisfying

$$\eta^{(0)}(a) = \eta^{(0)}(b) = \theta.$$

To show that $\eta^{(0)}$ is of class c' that is $\eta^{(0)}$ is in U, observe that by differentiation of (80)

$$g_{iy_j}\eta_j^{(0)} + g_{iy_j'}\eta_j^{(0)\prime} = \int_a^x \delta h_i\,dt + \frac{1}{b-a}\int_a^b \left[\delta g_i - \int_a^t \delta h_i\,ds\right]dt. \tag{81}$$

Since $|g_{iy_j'}| \neq 0$, (81) can be solved for $\eta^{(0)\prime}$. As all other quantities involved in (81) are continuous, $\eta^{(0)\prime}$ must have the same property.

By the hypothesis of nonconjugate end points the only element of U satisfying (80) is the zero element. Therefore $\eta^{(0)} \equiv \theta$ and (78) and (79) may be amended to read

$$u_i'(x,y^{(\varphi)};\eta^{(\varphi)}) \to u_i'(x,y^{(0)};\eta^{(0)}) = 0 \text{ weakly in } L^2 \quad (i=1,2,\ldots,n). \tag{82}$$

$$v_i'(x,y^{(\varphi)};\eta^{(\varphi)}) \to v_i'(x,y^{(0)};\eta^{(0)}) = 0 \text{ uniformly on } a \leq x \leq b \quad (i=1,2,\ldots,n). \tag{83}$$

Consider

$$\lim_{\varphi\to\infty}\int_a^b \delta T_i'(x,y^{(\varphi)};\eta^{(\varphi)})\delta T_i'(x,y^{(\varphi)};\eta^{(\varphi)})dx = \lim_{\varphi\to\infty}\int_a^b u_i^{(\varphi)\prime}u_i^{(\varphi)\prime} - 2u_i^{(\varphi)\prime}v_i^{(\varphi)\prime} + v_i^{(\varphi)\prime}v_i^{(\varphi)\prime})dx. \tag{84}$$

Although we have not previously defined all the notations in (84) their meanings are obvious. By (82) and (83) the terms involving v_i' may be disre-

garded. By lemma 15

$$\lim_{q\to\infty}\int_a^b u_i^{(q)\prime}u_k^{(q)\prime}dx=$$

$$\lim_{q\to\infty}\Big[\int_a^b \delta g_i^{(q)}\delta g_i^{(q)}dx-\frac{1}{b-a}\Big(\int_a^b \delta g_i^{(q)}dx\Big)\Big(\int_a^b \delta g_i^{(q)}dx\Big)\Big]$$

$$=\lim_{q\to\infty}\int_a^b \delta g_i^{(q)}\delta g_i^{(q)}dx.$$

Multiplying out, one obtains after dropping the superscript (q) from the derivatives of g_i.

$$\lim_{q\to\infty}\int_a^b \delta g_i\delta g_i dx$$

$$=\lim_{q\to\infty}\int_a^b (g_{iy_j}\eta_i^{(q)}g_{iy_k}\eta_k^{(q)}+2g_{iy_j}\eta_i^{(q)}g_{iy_k'}\eta_k^{(q)\prime}+$$

$$g_{iy_i'}\eta_i^{(q)\prime}g_{iy_k'}\eta_k^{(q)\prime})dx$$

$$=\lim_{q\to\infty}\int_a^b g_{iy_i'}g_{iy_k'}\eta_i^{(q)\prime}\eta_k^{(q)\prime}dx\equiv\lim_{q\to\infty}\int_a^b G_{ik}^{(q)}\eta_i^{(q)\prime}\eta_k^{(q)\prime}dx$$

since all terms involving $\eta^{(q)}$ go to zero.

Let $(\widetilde{A_{ij}})$ denote the transpose of the matrix (A_{ij}). Then (since by hypothesis $\det\left(g_{iy_j'}^{(q)}\right)\neq 0$).

$$\det(G_{ik}^{(q)})=\det\left(g_{iy_j'}^{(q)}\right)\det\left(\widetilde{g_{iy_k'}^{(q)}}\right)=\left[\det\left(g_{iy_j'}^{(q)}\right)\right]^2>0.$$

Hence $G_{ik}^{(q)}$ is a real, symmetric, positive definite matrix for each $q=1,2,\ldots$. Therefore, it follows that

$$\lim_{q\to\infty}\int_a^b \delta T_i'(x,y^{(q)};\eta^{(q)})\delta T_i'(x,y^{(q)};\eta^{(q)})dx=$$

$$\lim_{q\to\infty}\int_a^b G_{ik}^{(q)}\eta_i^{(q)\prime}\eta_k^{(q)\prime}dx\geq C>0.$$

However, (63) implies that this last limit is identically zero. Consequently, our assumption leads to a contradiction. Therefore there must exist at least one neighborhood of $y^{(0)}$ in which all elements have nonconjugate end points.

11. A Solution of the Fixed End Point Problem

To conclude the discussion of Newton's method, we wish to point out that with certain additional hypothesis we can make use of our previous results to obtain a solution of the fixed end point problem of the Calculus of Variations.

Let $f(x,y,y')$ be the integrand of the integral (1). A *strengthened Legendre condition* is said to hold in the region R if

$$f_{y_i'y_j'}(x,y,y')\pi_i\pi_j>0 \tag{85}$$

for all (x,y,y') in R and all $(\pi)\neq 0$. This implies that the determinant $|f_{y_iy_j}|\neq 0$ in R and is thus stronger than our previous condition that the determinant $|g_{iy_j'}|$ never vanishes in R.

The following terminology is well known and will be helpful. An arc y is said to satisfy a *strengthened Jacobi condition* if it has nonconjugate end points and has no subarc defined on $a\leq x\leq\xi<b$ which has conjugate end points.

Let (2) be replaced by the Euler equation (13). Let the region R be convex in y' and let the strengthened Legendre condition hold everywhere in R. The following theorem is then true.

THEOREM 6. *Let $y^{(0)}$ be an admissible element of Y having nonconjugate end points. Then if there exists a sufficiently small constant $D\geq 0$ such that $\|T(x,y^{(0)})\|<D$, the sequence of estimates derived from Newton's method will converge quadratically to a unique arc $\bar{y}$ of class c^2 which lies within a predetermined neighborhood of $y^{(0)}$, joins the end points of $y^{(0)}$, and satisfies the Euler equations. Moreover, if $\bar{y}$ is such that no interior point of $a\leq x\leq b$ is conjugate to $x=a$, $\bar{y}$ will afford a strong relative minimum to $I(y)$.*

By theorem 4, Newton's method converges quadratically to an arc $\bar{y}$ of class c^2, which joins the end points of $y^{(0)}$ and satisfies the Euler equations. The arc $\bar{y}$ also has the property of nonconjugate end points. This follows from lemma 14, which implies that the constant α_0 mentioned in the proof of theorem 4 can be chosen to have the additional property that no element of $(y^{(0)})_{\alpha_0}$ has conjugate end points. Since by hypothesis no subarc of $\bar{y}$ with initial point at $x=a$ has conjugate end points, a strengthened Jacobi condition is satisfied along $\bar{y}$. As it has also been assumed that a strengthened Legendre condition holds throughout R and that R is convex in y', we may conclude (see Bliss [2, p. 42, corollary 16.1]) that $\bar{y}$ affords a strong relative minimum to $I(y)$.

The following theorem gives further criteria that insure the convergence of the sequence of estimates derived from Newton's method to a solution of the fixed end point problem.

THEOREM 7. *Let a minimum y^* exist and have nonconjugate end points. Then there exists a constant $\alpha>0$ such that for any initial estimate $y^{(0)}$ taken from $(y^*)_\alpha$ the conclusions of theorem 6 hold.*

Upon recalling that a necessary condition for y^* to be a minimum is $T(x;y^*)=\theta$, one proceeds with the proof just as in theorem 5.

12. A Second Method for the Effective Solution of the Fixed End Point Problem

In section 2 it was remarked that in the application of Newton's method, one must solve the Euler equations of each term of a sequence of integrals whose integrands are quadratic in y and y'. Thus it is not inappropriate to introduce at this point a second method for solving fixed end point problems which will prove to be particularly applicable to the case of a quadratic integrand. In an actual computation a combination of Newton's method, and the method to be presented below, might prove to be advantageous.

Henceforth the symbol Y will denote the class of n-tuples of real valued functions on $a \leq x \leq b$ which have Lebesgue square integrable derivatives. Use was made of this class of functions in section 10. The symbol U will again denote the subclass of elements of Y, which vanish at the end points of $a \leq x \leq b$. Throughout the remainder of the paper we assume that all functions of Y are admissible and that the underlying region R is convex in y and y'. Our previous assumptions as to the nature of the integrand $f(x,y,y')$ are retained. Moreover, we assume that f and its derivatives are integrable along arcs in Y.

Assume that after starting from some initial estimate $y^{(0)}$ in Y one has obtained an "improved" estimate of a solution to the Euler equations $y^{(i)}$ also in Y. He then chooses the $i+1$st estimate to be

$$y^{(i+1)} = y^{(i)} - \alpha_i \eta^{(i)} \qquad \text{(no sum on } i\text{)}, \tag{86}$$

where $\eta^{(i)}$ is an element of U determined by the equation

$$\eta^{(i)} = T(x; y^{(i)}). \tag{87}$$

Here T is formed for the Euler equations as illustrated in section 2, and α_i is a real number.

A procedure similar to that just described has been announced by L. V. Kantorovitch [5] for the case of an integral whose integrand is quadratic in y and y'. However, his paper presents no proofs. Curry [6] has described an analogous process that can be used to minimize a function of n real variables. Other references may be found in the last paper cited.

In order to consider the convergence of the sequence defined by (86) it is helpful to examine the expression $I(y) - I(y - \alpha\eta)$. We may state the following lemma.

LEMMA 16. *Let y be an element of Y and let η be given by (87). Then,*

$$I(y) - I(y - \alpha\eta) = \alpha \int_a^b \eta_i' \eta_i' dx - \alpha^2 \int_a^b W dx \tag{88}$$

where

$$W \equiv A_{ij}\eta_i\eta_j + 2B_{ij}\eta_i\eta_i' + C_{ij}\eta_i'\eta_i'$$

$$A_{ij} = \int_0^1 (1-\mu) f_{y_iy_j}(x, y - \alpha\mu\eta, y' - \alpha\mu\eta')\, d\mu$$

and B_{ij}, C_{ij} are obtained from the formula for A_{ij} by replacing $f_{y_iy_j}$ by $f_{y_iy_j'}$ and $f_{y_i'y_j'}$ respectively.

An application of Taylor's theorem results in

$$I(y) - I(y = \alpha\eta)$$

$$= \alpha \int_a^b [f_{y_i}(x,y,y')\eta_i + f_{y_i'}(x,y,y')\eta_i']\, dx - \alpha^2 \int_a^b W dx.$$

By integration by parts

$$\int_a^b f_{y_i}\eta_i dx = -\int_a^b \left[\int_a^x f_{y_i} dt + C_i\right] \eta_i' dx;$$

and since

$$\int_a^b C_i \eta_i' dx = C_i[\eta_i(b) - \eta_i(a)] = 0,$$

the choice of C_i is arbitrary. Accordingly, we choose

$$C_i = \frac{1}{b-a} \int_a^b \left[f_{y_i}(t,y,y') - \int_a^t f_{y_i}(s,y,y')ds \right] dt.$$

Thus one may write

$$I(y) - I(y - \alpha\eta) = \alpha \int_a^b T_i' \eta_i' dx - \alpha_2 \int_a^b W dx$$

$$= \alpha \int_a^b \eta_i' \eta_i' dx - \alpha^2 \int_a^b W dx.$$

The next lemma follows directly from formula (88).

LEMMA 17. *Assume that for every number ν there exists a number $K > 0$ and a number α^* such that if $I(y) \leq \nu$ and $0 < \alpha \leq \alpha^*$ then*

$$\int_a^b W dx \leq K \int_a^b \eta_i' \eta_i' dx. \tag{89}$$

Then in case $I(y) \leq \nu$ and $0 < \alpha \leq \alpha^*$ one has the relation

$$I(y) - I(y - \alpha\eta) \geq \alpha(1 - \alpha K) \int_a^b \eta_i' \eta_i' dx. \tag{90}$$

Formula (90) helps to prove the following theorem.

THEOREM 8. *Let there exist finite numbers ν, $K > 0$, α^* such that if $I(y) \leq \nu$ and $0 < \alpha \leq \alpha^*$, formula (89) holds. Let $y^{(0)}$ be such that $I(y^{(0)}) \leq \nu$ and let α and $\bar{\alpha}$ be chosen so that $0 < \bar{\alpha} \leq \alpha < 1/K$. Then if $I(y)$ is bounded below on the class of arcs in Y joining the end points of $y^{(0)}$, the sequence of estimates $\{y^{(i)}\}$ defined by (86) and (87) is such that $\{I(y^{(i)})\}$ is convergent. Furthermore,*

$$T(x; y^{(i)}) \to \theta \quad \text{uniformly on } a \leq x \leq b \tag{91}$$

$$T'(x, y^{(i)}) \to \theta \quad \text{in } L^2. \tag{92}$$

That α may be chosen as specified in the hypothesis is a consequence of the fact that K is finite. As a result of such a choice of α, one obtains the inequality

$$I(y^{(0)}) > I(y^{(0)} - \alpha_0 \eta^{(0)}) \equiv I(y^{(1)})$$

from (90) by setting $y \equiv y^{(0)}$, $\alpha \equiv \alpha_0$. Since $I(y^{(1)}) < \nu$, the argument can be repeated provided $\eta^{(i)} \not\equiv \theta$. Hence a sequence $\{y^{(i)}\}$ such that

$$I(y^{(i)}) > I(y^{(i+1)}) \qquad (i = 0, 1, 2, \ldots)$$

is constructed. Due to the hypothesis that $I(y)$ is bounded below the sequence of definite integrals converges.

By (90) and the way $\{\alpha_i\}$ is chosen

$$I(y^{(i)})-I(y^{(i+1)})\geq\alpha_i(1-\alpha_iK)\int_a^b\eta_i^{(i)\prime}\eta_i^{(i)\prime}dx \tag{93}$$
$$\geq\bar{\alpha}(1-K\sup\alpha_i)\int_a^b\eta_i^{(i)\prime}\eta_i^{(i)\prime}dx\geq 0,$$

where here and in similar expressions that occur below there is no sum on the index that denotes which step of the iteration process we are considering. As the left-hand side of (93) can be made as small as desired by choosing i sufficiently large, we may conclude that

$$\lim_{i\to\infty}\left(\int_a^b\eta_i^{(i)\prime}\eta_i^{(i)\prime}dx\right)^{\frac{1}{2}}=0. \tag{94}$$

Conclusion (92) follows at once from (94). Since (91) is a consequence of (92), the theorem is proved.

The problem of determining general conditions under which an assumption of the type made in lemma 17 is valid seems to be a difficult one. However, special cases may be considered in which K is independent of ν and α^* is arbitrary. A rather trivial example of this occurs when the integrand is given by

$$f(x,y,y')=g(x)\sqrt{1+y'^2},\qquad g(x)\geq 0.$$

If $2K\geq|g(x)|$ on $a\leq x\leq b$, we have

$$\int_a^b Wdx=\int_a^b\int_0^1(1-\mu)\frac{g(x)\eta'^2}{[1+(y'-\alpha\mu\eta')^2]^{\frac{3}{2}}}\,d\mu dx$$
$$\leq K\int_a^b\eta'^2dx.$$

A much more important case is that in which the integrand is quadratic, that is,

$$f(x,y,y')=A_{ij}(x)y_iy_j+2B_{ij}(x)y_iy_j'+C_{ij}(x)y_i'y_j'+$$
$$D_i(x)y_i+E_i(x)y_i'+F_i(x). \tag{95}$$

For $f(x,y,y')$ given by (95) we obtain upon integrating out μ

$$\int_a^b Wdx=\int_a^b[A_{ij}(x)\eta_i\eta_j+B_{ij}(x)\eta_i\eta'_j+C_{ij}(x)\eta_i'\eta_i']dx$$
$$\leq K\int_a^b\eta_i'\eta_i'dx$$

for a suitably chosen constant K provided $\eta_i(a)=\eta_i(b)=0$. This last result follows from the Schwarz inequality and the fact that the integrand is a quadratic form in η and η' having continuous coefficients.

By setting $y\equiv y^{(0)}$, $\eta\equiv\bar{y}-y^{(0)}$, $\alpha\equiv-1$ in the computations employed to establish lemma 16, one obtains the formula

$$I(\bar{y})=I(y^{(0)})+\int_a^b T_i'(x;y^{(0)})\eta_i'dx+\int_a^b Wdx, \tag{96}$$

which is true for all $\bar{y}$ in Y which join the end points of $y^{(0)}$. This equation is helpful in the proofs to be given below.

LEMMA 18. *Let there exist a constant $k>0$ such that*

$$\int_a^b Wdx\geq k\int_a^b\eta_i'\eta_i'dx \tag{97}$$

for all η in U and all y in Y. Then $I(y)$ is bounded below on the class of elements of Y joining the end points of some arbitrary fixed $y^{(0)}$ in Y. Furthermore, given a constant ν there exists a constant M such that for each y in Y joining the end points of $y^{(0)}$ for which $I(y)\leq\nu$

$$\int_a^b\eta_i'\eta_i'dx\leq M. \tag{98}$$

In (98) we have $\eta\equiv y-y^{(0)}$.

By Schwarz' inequality

$$\int_a^b T_i'(x;y^{(0)})\eta_i'\leq$$
$$\left(\int_a^b T_i'(x;y^{(0)})T_i'(x;y^{(0)})dx\right)^{\frac{1}{2}}\left(\int_a^b\eta_i'\eta_i'dx\right)^{\frac{1}{2}}$$
$$\equiv h\left(\int_a^b\eta_i'\eta_i'dx\right)^{\frac{1}{2}}.$$

Thus by (96) and (97)

$$I(y)\geq I(y^{(0)})-h\left(\int_a^b\eta_i'\eta_i'dx\right)^{\frac{1}{2}}+k\int_a^b\eta_i'\eta_i'dx$$
$$=I(y^{(0)})-\frac{h^2}{4k}+\left[\frac{h}{2\sqrt{k}}-\left(k\int_a^b\eta_i'\eta_i'dx\right)^{\frac{1}{2}}\right]^2$$
$$\geq I(y^{(0)})-\frac{h^2}{4k},$$

provided y joins the end points of $y^{(0)}$. This proves that $I(y)$ is bounded below on the class of functions joining the end points of $y^{(0)}$. The remainder of the lemma is also a ready consequence of this last inequality.

Condition (97) does not seem to be very widely applicable. However, in case the integrand is quadratic the strengthened Jacobi condition in conjunction with the strengthened Legendre condition implies the existence of a constant $k>0$ such that (97) holds.

LEMMA 19. *Let $\{y^{(q)}\}$ be a sequence in Y such that*

$$y_i^{(q)}\to\bar{y}_i \text{ uniformly on } a\leq x\leq b,\quad (i=1,2,\ldots,n) \tag{99}$$
$$y_i^{(q)\prime}\to\bar{y}_i' \text{ almost uniformly}\quad (i=1,2,\ldots,n) \tag{100}$$
$$T_i'(x;y^{(q)})\to 0 \text{ almost uniformly}\quad (i=1,2,\ldots,n). \tag{101}$$

Then

$$T_i'(x;\bar{y})=0 \text{ almost everywhere}\quad (i=1,2,\ldots,n). \tag{102}$$

Let $\boldsymbol{I}$ denote the interval $a \leq x \leq b$. Since $\overline{y}'$ is in L^2 it is summable and hence finite almost everywhere. Hence given $\epsilon > 0$ there exists by (100) a set S_1 of measure less than $\epsilon/2$ such that $y_i^{(\varphi)\prime} \to \overline{y}_i'$ uniformly on $\boldsymbol{I} - S_1$ for $(i = 1,2,\ldots n)$. Of course, S_1 must contain all the infinites of $\overline{y}'$. The summability of $\overline{y}'$ also implies the existence of a set S_2 of measure less than $\epsilon/4$ such that $\overline{y}_i'$ is bounded on $\boldsymbol{I} - S_2$ for $(i = 1,2,\ldots,n)$. Therefore, if S_3 is an open set of measure less than ϵ containing $S_1 + S_2$, the following statements are valid:

(a) $\boldsymbol{I} - S_3$ is closed and bounded,
(b) $\overline{y}_i'$ is bounded on $\boldsymbol{I} - S_3$ $(i = 1,2,\ldots,n)$,
(c) $y_i^{(\varphi)} \to \overline{y}_i$ uniformly on $\boldsymbol{I} - S_3$ $(i = 1,2,\ldots,n)$,
(d) $y_i^{(\varphi)\prime} \to \overline{y}_i'$ uniformly on $\boldsymbol{I} - S_3$ $(i = 1,2,\ldots,n)$.

By (*a*) through (*d*) it can be concluded that

$$f_{y_i'}(x, y^{(\varphi)}(y^{(\varphi)\prime}) \to f_{y_i'}(x, \overline{y}, \overline{y}) \quad \text{uniformly on } \boldsymbol{I} - S_3$$

and similarly for f_{y_i}. Hence it readily is verified that $T_i'(x; y^{(\varphi)}) \to T_i'(x, \overline{y})$ uniformly on $\boldsymbol{I} - S_3$ for $(i = 1,2,\ldots,n)$, that is, almost uniformly and thus almost everywhere. Conclusion (102) is then obtainable from (101).

If one wishes, he may express (102) in the form $T(x; \overline{y}) = \theta$. This follows directly by integration.

LEMMA 20. *Let $\overline{y}$ be an element of Y satisfying (102). Then if there exists $k > 0$ such that (97) holds, $\overline{y}$ affords a unique minimum to $I(y)$ on the class of functions in Y which join the end points of $\overline{y}$.*

Let y be any element of Y joining the end points of $\overline{y}$. By (96)

$$I(y) - I(\overline{y}) = \int_a^b T_i'(x; \overline{y})(y_i' - \overline{y}_i')\,dx + \int_a^b W\,dx.$$

By hypothesis $T_i'(x; \overline{y}) = 0$ almost everywhere and $y_i' - \overline{y}_i'$ is in L^2. Thus the first integral on the right is zero by Hölder's inequality. Therefore, if $y \neq \overline{y}$,

$$I(y) - I(\overline{y}) = \int_a^b W\,dx \geq k \int_a^b (y_i' - \overline{y}_i')(y_i' - \overline{y}_i')\,dx > 0.$$

Consequently,

$$I(y) > I(\overline{y}). \tag{103}$$

Assume that y also satisfies (102). Then by reversing the roles of y and $\overline{y}$ one obtains $I(\overline{y}) > I(y)$. This contradicts (103), and so implies that the assumption that $y \neq \overline{y}$ satisfies (102) is false. Hence y cannot be a minimum.

THEOREM 9. *Let there exist finite numbers ν, $K > 0$, α^* such that if $I(y) \leq \nu$ and $0 < \alpha < \alpha^* < 1/K$, formula (89) holds. Let $y^{(0)}$ be such that $I(y^{(0)}) \leq \nu$ and let α and $\overline{\alpha}$ be chosen so that $0 < \overline{\alpha} \leq \alpha < \alpha^*$. Then if there exists a constant $k > 0$ such that (97) is satisfied, the conclusions of theorem 8 are satisfied. Furthermore, there exists a function $\overline{y}$ such that for the sequence $\{y^{(i)}\}$ described in theorem 8 we have*

$$y^{(i)} \to \overline{y} \quad \text{uniformly on } a \leq x \leq b$$

$$y^{(i)\prime} \to \overline{y}' \quad \text{in the mean of order 2.}$$

The limit $\overline{y}$ affords a unique minimum to $I(y)$ on the class of functions in Y joining the end points of $y^{(0)}$.

By lemma 18, $I(y)$ is bounded below on the class of y in Y joining the end points of $y^{(0)}$. Since all of the other hypotheses of theorem 8 have been assumed, the conclusions of that theorem must be valid.

As the sequence $\{y^{(i)}\}$ of theorem 8 is such that each $y^{(i)}$ joins the end points of $y^{(0)}$ and

$$I(y^{(i)}) < I(y^{(0)}) \leq \nu \qquad (i = 1,2,\ldots),$$

formula (98) of lemma 18 is applicable to each term. This formula implies the existence of $\overline{M} \geq 0$ such that for all n, m

$$\left(\int_a^b \eta_i^{(n,m)\prime}\, \eta_i^{(n,m)\prime}\, dx \right)^{\frac{1}{2}} \leq \overline{M}, \tag{104}$$

where $\eta^{(n,m)} \equiv y^{(n)} - y^{(m)}$. For $n > m$, $I(y^{(n)}) - I(y^{(m)})$ is negative. Thus from substitution in the main formula of the proof of lemma 18 it follows that

$$0 \leq k \int_a^b \eta_i^{(n,m)\prime}\, \eta_i^{(n,m)\prime}\, dx \leq h_m \left(\int_a^b \eta_i^{(n,m)\prime}\, \eta_i^{(n,m)\prime}\, dx \right)^{\frac{1}{2}}, \qquad n > m,$$

where

$$h_m \equiv \left(\int_a^b T_i'(x; y^{(m)})\, T_i'(x; y^{(m)})\, dx \right)^{\frac{1}{2}} \to 0, \qquad m \to \infty.$$

Consequently, the existence of $\overline{y}$ such that $y^{(i)\prime} \to \overline{y}'$ in the mean of order 2 follows from 104. This last result implies $y^{(i)} \to \overline{y}$ uniformly on $a \leq x \leq b$. By recalling (92) and taking subsequences one finally obtains a subsequence $\{y^{(\varphi)}\}$ such that

$$y^{(\varphi)} \to \overline{y} \quad \text{uniformly on } a \leq x \leq b$$

$$y^{(\varphi)\prime} \to \overline{y}' \quad \text{almost uniformly}$$

$$T'(x; y^{(\varphi)}) \to \theta \quad \text{almost uniformly.}$$

Consequently (by lemma 19) $\overline{y}$ satisfies (102). That $\overline{y}$ is unique is a consequence of lemma 20. Hence the theorem is proved. It is not difficult to show that $\overline{y}(x)$ is of class c^2.

It is clear that these results apply to the case when f is a polynomial of degree two in y_i'.

13. An Example of Numerical Computation

Consider the problem of minimizing the integral

$$I(y) = \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\, dx, \qquad y > 0,$$

whose Euler equation is

$$y''=\frac{1+y'^2}{y}. \tag{105}$$

To apply Newton's method write (105) as

$$T=1+y'^2-yy''=0.$$

Then

$$\delta T=2y'\eta'-y''\eta-y\eta'',$$

and the differential equation $T+\delta T=0$ can be put into the form

$$\eta''=A\eta+B\eta'+C,$$

where

$$A=\frac{y''}{y},\quad B=\frac{2y'}{y},\quad C=\frac{1+y'^2}{y}-y''.$$

The general solution of (105) is given by

$$y=b\cosh\frac{x-a}{b}. \tag{106}$$

Given two points (x_1, y_1) and (x_2, y_2) three cases can occur.

(1) Two catenaries (106) join the given points. In this case one will be minimizing and the other will not.

(2) One catenary (106) joins the given points. This catenary is not minimizing.

(3) No catenary (106) joins the given points. The examples for computation were selected with this in mind.

EXAMPLE I:

The points (x_1, y_1), (x_2, y_2) are respectively $(-2, 3.086)$ and $(2, 3.086)$. The solutions are

$$y=2\cosh\frac{x}{2}$$

$$y=1.404\cosh\frac{x}{1.404},$$

where the first is minimizing and the second is not. The results of computation are presented below in tables 1, 2, and 3. In these tables as well in tables 4, 5, y denotes the exact solution, $y^{(0)}$ the initial approximation, and $y^{(i)}$ the approximation at the ith step. The symbol c_i denotes the value of

$$c=\frac{1+y'^2}{y}-y''$$

along the curve $y^{(i)}$. Along a solution of (105) we have $c=0$. The function v_i found in tables 2 to 5 is a solution of the equation $\delta c=0$, which results from taking the variation of the equation $c_i=0$. The evaluation of v_i can be arranged to be a by-product of the computation of $y^{(i+1)}$. It follows from the theory of conjugate points that y will be a minimizing arc if for c_i sufficiently small v_i does not vanish more than once. If v_i vanishes twice, y is not minimizing.

It is of interest to note that convergence to a solution of the Euler equations that was not a minimum was obtained. This shows the necessity of making additional assumptions (as was done in section 11) in order to insure convergence to a minimum.

EXAMPLE II:

The points (x_1, y_1), (x_2, y_2), are, respectively, (0, 2.5894) and (5, 5.9284). The solutions are

$$y=2\cosh\frac{x-1.5}{2}$$

$$y=1.8883\cosh\frac{x-1.5806}{1.8883}.$$

The first arc is minimizing and the second is not. It will be noticed that although the two solutions are quite close to each other convergence to each one was obtained.

EXAMPLE III:

The points (x_1,y_1) and (x_2,y_2) are, respectively, (0,2), (5,3). In this case there is no solution. After an initial estimate consisting of the line joining (0,2) and (5,3), the second iteration resulted in points (x,y) lying outside of the domain in which $y>0$.

The material in this section was supplied to the author by M. R. Hestenes, under whose auspices the computations were carried out.

TABLE 1

$y=2\cosh x/2$, $y^{(0)}=3.086$ $\quad(-2\leq x\leq 2)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	$y^{(4)}$	y	c_0	c_1	c_2	c_3	c_4
0	3.086	2.438	2.132	2.015	1.994	2.000	.324	.086	.018	.002	.000
±.2	3.086	2.447	2.140	2.026	2.004	2.010	.324	.087	.019	.002	.000
±.4	3.086	2.464	2.167	2.056	2.034	2.040	.324	.087	.020	.003	.000
±.6	3.086	2.497	2.213	2.106	2.085	2.090	.324	.092	.021	.003	.000
±.8	3.086	2.542	2.277	2.177	2.157	2.162	.324	.096	.023	.003	.000
±1.0	3.086	2.600	2.360	2.269	2.251	2.255	.324	.101	.025	.004	.000
±1.2	3.086	2.672	2.463	2.383	2.367	2.371	.324	.107	.028	.004	.000
±1.4	3.086	2.756	2.586	2.520	2.507	2.510	.324	.114	.031	.005	.000
±1.6	3.086	2.853	2.730	2.682	2.673	2.675	.324	.121	.034	.006	.000
±1.8	3.086	2.963	2.897	2.870	2.865	2.866	.324	.128	.038	.006	.000
±2.0	3.086	3.086	3.086	3.086	3.086	3.086	.324	.136	.042	.007	.000

TABLE 2

$y=2\cosh(x/2)$, $y^{(0)}=1.886+.3x^2$ $\quad(-\leq x\leq 2)$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	y	c_0	c_1	c_2	v_1
0	1.886	2.053	2.003	2.000	−.070	.010	.001	1.000
±.2	1.898	2.063	2.013	2.010	−.066	.010	.001	.995
±.4	1.934	2.092	2.043	2.040	−.053	.010	.001	.981
±.6	1.994	2.140	2.093	2.091	−.034	.009	.001	.957
±.8	2.078	2.208	2.165	2.162	−.008	.009	.001	.923
±1.0	2.186	2.297	2.258	2.255	−.022	.008	.001	.877
±1.2	2.318	2.407	2.373	2.371	.055	.007	.001	.818
±1.4	2.474	2.540	2.512	2.510	.089	.006	.001	.745
±1.6	2.654	2.696	2.676	2.675	.124	.005	.001	.656
±1.8	2.858	2.878	2.867	2.866	.158	.003	.001	.548
±2.0	3.086	3.086	3.086	3.086	.191	.002	.001	.419

TABLE 3

$$y = 1.401 \cosh \frac{x}{1.404}, \quad y^{(0)} = 1.286 + .45x^2 \qquad (-2 \le x \le 2)$$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	y	c_0	c_1	c_2	v_2
0	1.286	1.397	1.407	1.404	−.123	.010	.000	1.000
± .2	1.304	1.381	1.421	1.418	−.108	.010	.000	.989
± .4	1.358	1.425	1.464	1.462	−.068	.010	.000	.957
± .6	1.448	1.499	1.537	1.534	−.008	.009	.000	.900
± .8	1.574	1.604	1.640	1.638	.065	.008	.000	.816
±1.0	1.736	1.744	1.778	1.776	.143	.007	.001	.697
±1.2	1.934	1.921	1.951	1.949	.220	.006	.001	.542
±1.4	2.168	2.138	2.163	2.162	.293	.005	.001	.337
±1.6	2.438	2.401	2.420	2.419	.361	.003	.001	.073
±1.8	2.744	2.715	2.725	2.725	.421	.009	.001	−.265
±2.0	3.086	3.086	3.086	3.086	.474	.017	.001	−.693

TABLE 4

$$y = 2 \cosh \frac{x-1.5}{2}, \quad y^{(0)} = 2.5894 - .8322x + .3x^2 \qquad (0 \le x \le 5)$$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	y	c_0	c_1	c_2	c_3	v_3
.0	2.589	2.589	2.589	2.589	2.589	.054	.012	.004	.001	.000
.2	2.435	2.498	2.449	2.439	2.438	.019	.011	.004	.001	.189
.4	2.304	2.398	2.331	2.313	2.310	−.014	.011	.004	.001	.357
.6	2.198	2.287	2.235	2.210	2.206	−.044	.011	.003	.001	.507
.8	2.116	2.226	2.160	2.128	2.124	−.069	.010	.003	.001	.642
1.0	2.057	2.184	2.106	2.068	2.062	−.088	.010	.003	.001	.762
1.2	2.023	2.160	2.072	2.029	2.022	−.099	.010	.003	.001	.871
1.4	2.012	2.154	2.057	2.010	2.002	−.103	.005	.002	.000	.970
1.6	2.026	2.167	2.061	2.010	2.002	−.098	.000	.002	.000	1.058
1.8	2.063	2.198	2.085	2.031	2.022	−.086	.009	.002	.000	1.138
2.0	2.125	2.248	2.129	2.072	2.062	−.066	.009	.002	.000	1.209
2.2	2.211	2.316	2.192	2.133	2.124	−.040	.009	.002	.000	1.272
2.4	2.320	2.404	2.277	2.215	2.206	−.010	.009	.002	.000	1.328
2.6	2.454	2.513	2.383	2.320	2.310	.023	.009	.002	.000	1.374
2.8	2.611	2.643	2.511	2.448	2.438	.058	.008	.002	.000	1.412
3.0	2.793	2.795	2.663	2.589	2.590	.093	.008	.002	.000	1.441
3.2	2.998	2.970	2.840	2.777	2.788	.128	.008	.002	.000	1.459
3.4	3.221	3.171	3.043	2.982	2.972	.162	.007	.002	.001	1.464
3.6	3.482	3.398	3.276	3.217	3.208	.194	.007	.002	.000	1.456
3.8	3.759	3.654	3.539	3.483	3.474	.224	.007	.002	.001	1.431
4.0	4.061	3.941	3.836	3.785	3.776	.252	.008	.003	.001	1.388
4.2	4.386	4.261	4.169	4.124	4.116	.277	.009	.003	.001	1.323
4.4	4.736	4.616	4.541	4.503	4.428	.300	.011	.004	.001	1.232
4.6	5.109	5.011	4.955	4.928	4.924	.323	.013	.004	.001	1.111
4.8	5.507	5.447	5.416	5.401	5.400	.343	.017	.004	.001	.956
5.0	5.928	5.928	5.928	5.928	5.928	.361	.022	.005	.001	.769

TABLE 5

$$y = 1.888 \cosh \frac{x-1.581}{1.888}, \quad y^{(0)} = 2.5894 - 1.2322x + .38x^2 \qquad (0 \le x \le 5)$$

x	$y^{(0)}$	$y^{(1)}$	$y^{(2)}$	$y^{(3)}$	y	c_0	c_1	c_2	c_3	v_3
0	2.589	2.589	2.589	2.589	2.589	.212	.002	.000	.002	.000
.2	2.398	2.375	2.403	2.414	2.416	.169	−.008	.008	.001	.185
.4	2.157	2.194	2.245	2.267	2.270	.103	−.013	.007	.001	.344
.6	1.987	2.044	2.114	2.145	2.149	.046	.013	.006	.001	.479
.8	1.847	1.923	2.009	2.047	2.052	−.008	.008	.005	.001	.595
1.0	1.737	1.828	1.927	1.927	1.978	−.056	.012	.005	.001	.693
1.2	1.658	1.758	1.859	1.920	1.927	−.065	.016	.004	.001	.777
1.4	1.609	1.712	1.833	1.880	1.897	−.121	.019	.003	.001	.847
1.6	1.591	1.690	1.819	1.880	1.898	−.131	.021	.003	.001	.905
1.8	1.603	1.690	1.827	1.892	1.901	−.124	.022	.002	.001	.953
2.0	1.645	1.714	1.857	1.926	1.935	−.102	.022	.002	.001	.989
2.2	1.718	1.761	1.910	1.981	1.990	−.065	.021	.002	.001	1.015
2.4	1.821	1.831	1.986	2.059	2.068	−.018	.019	.002	.001	1.030
2.6	1.954	1.927	2.085	2.160	2.170	.035	.017	.002	.001	1.034
2.8	2.118	2.049	2.210	2.285	2.295	.091	.015	.002	.001	1.024
3.0	2.313	2.198	2.361	2.437	2.447	.147	.012	.003	.001	1.000
3.2	2.538	2.378	2.541	2.616	2.626	.201	.009	.003	.001	.960
3.4	2.793	2.590	2.751	2.824	2.834	.252	.006	.004	.001	.900
3.6	3.078	2.839	2.995	3.065	3.074	.300	.005	.004	.001	.818
3.8	3.394	3.126	3.274	3.340	3.349	.342	.004	.005	.001	.710
4.0	3.741	3.456	3.593	3.654	3.662	.381	.006	.006	.001	.570
4.2	4.117	3.834	3.955	4.008	4.015	.416	.009	.006	.001	.394
4.4	4.524	4.264	4.364	4.408	4.414	.447	.016	.007	.001	.174
4.6	4.962	4.732	4.826	4.858	4.862	.474	.025	.007	.001	−.096
4.8	5.430	5.305	5.345	5.363	5.365	.499	.038	.008	.002	−.426
5.0	5.925	5.928	5.928	5.928	5.928	.521	.053	.008	.002	−.826

14. References

In addition to the titles of the references made in the body of the paper, the following list contains the titles of various other articles whose contents are related to the work presented in this paper.

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